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Generalized Schwarzschild Metric Parameter

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Abstract: General Relativity is one of the most important branches of Physics that changes radically the concepts of space, time and matter. It can describe successfully the behavior of the gravitational field and it gives wonderful solutions of many gravitational problems, which agrees with astronomical observations to a high degree of precision. But unfortunately, it suffers from noticeable setbacks. For example, the cosmological model based on it suffers from certain cosmological problems. These problems include the behavior of quasars and pulsars, where the field is too strong and that is difficult to be understood in terms of General Relativity. Many attempts were made to go beyond General Relativity. Among these the Generalized Field Equation (GFE), which is based on a more general form of fourth order differential equations, looks more promising. This is due to the successes of it in making these problems controllable, in this work exact solution of Generalized Field Equation metric has been done where the field is strong near the black hole horizon and agree with General Relativity solution at weak field a distance from horizon, and apply this generalized metric in radial null geodesic and Eddington-Finkelstein Coordinates

Keywords: Schwarzschild Metric, General relativity (GR), generalized field equation (GFE), Einstein's theory, black holes,

Introduction

General relativity is one of the most important theories that have been used to describe the nature of the gravitational field. It is based on a linear Lagrangian, which can successfully describe the behavior of the universe and all astronomical objects.

In Einstein's theory of general relativity, the Schwarzschild metric (also known as the Schwarzschild solution) is an exact solution to the Einstein field equations that describes the gravitational field outside a spherical mass, assuming that the electric charge of the mass, angular momentum of the mass, and universal cosmological constant are all zero. The solution is a useful approximation for describing slowly rotating astronomical objects like stars and planets, including Earth and the Sun. This solution was found also by Karl Schwarzschild in 1916 [1], and around the same time independently by Johannes Droste, who published his more complete and modern-looking discussion four months after Schwarzschild.

According to Brinkhoff's theorem [6], the Schwarzschild metric is the most general spherically symmetric vacuum solution of the Einstein field equations. A Schwarzschild black hole or static black hole is a black hole that has neither electric charge nor angular momentum. A Schwarzschild black hole behavior is described by the Schwarzschild metric, and cannot be distinguished from any other Schwarzschild black holes except by its mass. The Schwarzschild black hole characterized by a surrounding spherical boundary, called the event horizon, which is situated at the Schwarzschild radius. This is also called the radius of a black hole. The boundary is not a physical surface, and a person who fell through the event horizon (before being torn apart by tidal forces), would not notice any physical surface at that position; it is a mathematical surface which is significant in determining the black hole's properties. Any non-rotating and non-charged mass that is smaller than its Schwarzschild radius forms a black hole. The solution of the Einstein field equations is valid for any mass M, so in principle a Schwarzschild black hole of any mass could exist if conditions became sufficiently favorable to allow for its formation. In the vicinity of a Schwarzschild black hole, space is highly deformed so much that even light rays are deflected, and nearby light can be deflected so much that it travels several times around the black hole.A new version of GR, by keeping its beautiful geometrical language and abandoning Newton Poisson equation. This new version is first pro- posed by Lanczos [8] and then by Ali Eltahir [9].In this paper, we will find exact solution of Generalized Metric of Gravitation [10].

Generalized Metric Solution

We are going to generalize the Schwarzschild metric to fit with strong gravity near the black holes where the ratio $\frac{2m}{r}$ ~1. Anyway, the Schwarzschild metric can be in the form

$$ds^2 = -e^{\mu}dt^2 + e^{\nu}dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\varphi^2 \quad (1)$$

Using the values for $g_{\mu\nu}$ from equation (1), we can calculate the nonzero Christoffel symbols

$$\Gamma^r_{rr} = \frac{A'(r)}{2A(r)} = \nu' \qquad \Gamma^r_{\theta\theta} = -r\, e^{2\nu}\Gamma^r_{\theta\theta} = -r sin^2\theta\, e^{2\nu}$$

$$\Gamma^r_{tt} = \frac{B'(r)}{2A(r)} = \mu' e^{2\mu-2\nu}\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}\Gamma^\theta_{\phi\phi} = -\sin\theta cos\theta$$

$$\Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{r}\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta\, \Gamma^t_{rt} = \Gamma^t_{tr} = \frac{B'(r)}{2B(r)} = \mu' \quad (2)$$

One of the possible solution for μ and ν as a function of r is

$$\nu = -\mu \quad \Longrightarrow \quad B = e^{-\frac{2m}{r}} \quad A = e^{\frac{2m}{r}} \quad (3)$$

This solution is more general and avoid coordinate singularity when $2m = r$, moreover it tends to GR solution for large values of r.That leads to Christoffel symbols in the form

$$\Gamma^r_{rr} = \frac{A'(r)}{2A(r)} = -\frac{m}{r^2}\Gamma^r_{\theta\theta} = -r\, e^{\frac{-2m}{r}}\Gamma^r_{\theta\theta} = -r sin^2\theta\, e^{\frac{-2m}{r}}$$

$$\Gamma^r_{tt} = \frac{B'(r)}{2A(r)} = \frac{m}{r^2}e^{\frac{-4m}{r}}\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}\Gamma^\theta_{\phi\phi} = -\sin\theta cos\theta$$

$$\Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{r}\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta\, \Gamma^t_{rt} = \Gamma^t_{tr} = \frac{B'(r)}{2B(r)} = \frac{m}{r^2} \quad (4)$$

The metric takes the form

$$ds^2 = -e^{-\frac{2m}{r}}dt^2 + e^{\frac{2m}{r}}dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\varphi^2 \quad (5)$$

The exponential function can take a form of a series

$$e^{-\frac{2m}{r}} = 1 - \frac{2m}{r} + \frac{m^2}{r^2} - \cdots \quad and \quad e^{\frac{2m}{r}} = \frac{1}{1 - \frac{2m}{r} + \frac{m^2}{r^2} - \cdots} \quad (6)$$

The weak field approximation for generalized metric takes place for large value of r where

$$e^{\frac{2m}{r}} \to \frac{1}{1-\frac{2m}{r}} e^{-\frac{2m}{r}} \to 1 - \frac{2m}{r} \quad (7)$$

For strong gravitational field near event horizon where $\frac{2m}{r}$ ~1 , in this case the Lagrangian take nonlinear value[8] , also the GFE [9,10,11] uses $L = -\beta R + \alpha R^2 + \gamma$ instead of GR. Lagrangian

Radial Null Geodesics for Generalized Metric

Let us search for the null geodesics of the generalized Schwarzschild metric that are radial, i.e., along which $\theta = const$ and $\varphi = const$. They are found by setting
$d\theta = const$ and $d\varphi = const$ thus

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = 0 \quad (8)$$

$$0 = -B(r)dt^2 + A(r)dr^2 \quad (9)$$

$$0 = -e^{-\frac{2m}{r}}dt^2 + e^{\frac{2m}{r}}dr^2 \quad (10)$$

$$dt^2 = \frac{dr^2}{\left(e^{-\frac{2m}{r}}\right)^2} \quad (11)$$

Hence, the radial null geodesics are governed by

$$dt = \pm\ e^{\frac{2m}{r}}dr \quad (12)$$

This equation is easily integrated

$$t = \pm r e^{\frac{2m}{r}} \mp 2m\, Ei\left(\frac{2m}{r}\right) + const \quad (13)$$

The $-$ and + signs represent ingoing and outgoing null geodesics where

$$t = r e^{\frac{2m}{r}} - 2m\, Ei\left(\frac{2m}{r}\right) + u \qquad outgoing \quad (14)$$

$$t = -re^{\frac{2m}{r}} + 2m\, Ei\left(\frac{2m}{r}\right) + v \qquad ingoing \quad (15)$$

Generalized Eddington-Finkelstein Coordinates

The parameter v introduced in Eq. (15) can be seen as a label for the ingoing radial null geodesics each of these curves is entirely identified by the data (v, θ, φ) which remains fixed along it. Let us promote v to a spacetime coordinate, instead of t, i.e., let us consider the coordinate system (v, r, θ, φ) with the relation to Schwarzschild-Droste coordinates (t, r, θ, φ)

$$v = t + re^{\frac{2m}{r}} - 2m\, Ei\left(\frac{2m}{r}\right) \qquad (16)$$

It follows immediately that

$$dv = dt + \frac{(r-2m)e^{\frac{2m}{r}}dr}{r} - \frac{2m\, e^{\frac{2m}{r}}dr}{r} \qquad (17)$$

$$dv = dt + \frac{(r-4m)e^{\frac{2m}{r}}dr}{r} \qquad (18)$$

i.e.

$$dt = dv - \frac{(r-4m)e^{\frac{2m}{r}}dr}{r} \qquad (19)$$

Taking the square gives

$$dt^2 = dv^2 - 2\frac{(r-4m)e^{\frac{2m}{r}}drdv}{r} + \frac{(r-4m)^2e^{\frac{4m}{r}}dr^2}{r^2} \qquad (20)$$

Substituting this expression dt^2in Eq. (6)yields the metric components with respect to the coordinates $(\tilde{x}^\alpha) := (v, r, \theta, \varphi)$

$$g_{\mu\nu}\, dx^\mu dx^\nu = -e^{-\frac{2m}{r}}dv^2 + 2\frac{(r-4m)\, drdv}{r} - \frac{(r-4m)^2e^{\frac{2m}{r}}dr^2}{r^2} + e^{\frac{2m}{r}}dr^2 + r^2(d\theta^2 + \theta d\varphi^2) \quad (21)$$

Using equation (16) add and subtract r thus

$$v = t + re^{\frac{2m}{r}} - 2m\, Ei\left(\frac{2m}{r}\right) - r + r \qquad (22)$$

One can get

$$\tilde{t} = t + re^{\frac{2m}{r}} - 2m\, Ei\left(\frac{2m}{r}\right) - r \qquad (23)$$

$$v = \tilde{t} + r \qquad \Leftrightarrow \qquad dv = d\tilde{t} + dr \qquad (24)$$

Substituting this expression for dv in Eq. (20)

$$\tilde{g}_{\mu\nu}\, dx^\mu dx^\nu = -e^{-\frac{2m}{r}}(d\tilde{t} + dr)^2 + 2\frac{(r-4m)\, dr(d\tilde{t} + dr)}{r} - \frac{(r-4m)^2e^{\frac{2m}{r}}dr^2}{r^2} + e^{\frac{2m}{r}}dr^2 + r^2(d\theta^2 + \theta d\varphi^2) \qquad (25)$$

Rearranging the terms one can get the generalized Eddington-Finkelstein metric in the form

$$\tilde{g}_{\mu\nu}\, dx^\mu dx^\nu = -e^{-\frac{2m}{r}}d\tilde{t}^2 + \left(2\frac{(r-4m)}{r} - 2e^{-\frac{2m}{r}}\right)drd\tilde{t} + \left(-\frac{(r-4m)^2e^{\frac{2m}{r}}}{r^2} + 2\frac{(r-4m)}{r} - e^{-\frac{2m}{r}} + e^{\frac{2m}{r}}\right)dr^2 + r^2(d\theta^2 + \theta d\varphi^2) \qquad (26)$$

One can make weak field approximation For $r \gg m$ that makes the metric (26) reduce to Eddington-Finkelstein metric[12] [13] which based on GR

$$\tilde{g}_{\mu\nu}\, dx^\mu dx^\nu = -\left(1 - \frac{2m}{r}\right)d\tilde{t}^2 + \frac{4m}{r}drd\tilde{t} + \left(1 + \frac{2m}{r}\right)dr^2 + r^2(d\theta^2 + \theta d\varphi^2) \qquad (27)$$

Conclusion and Discussion

The generalized metric in equation (5) convenient for strong field near black holes and tends to Schwarzschild metric at large values r, this metric can be used to reformulate the equations of general relativity to be valid for both strong and weak fields and is expected to solve many problems related to strong gravity. equation (13) gives radial null geodesics by using this metric and use to reformulate Eddington-Finkelstein metric eqution (26). The weak field approximations for large r always give the same results that set by GR

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