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Interlace Polynomials of A Special Type of Eulerian Graph

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Abstract: In this paper, we study the interlace polynomials of a type of Eulerian graph and other related graphs. Explicit formulas, special values, and behavior of the coefficients of these polynomials are provided. Some of the properties are applied to describe the studied graphs.
Keywords: Interlace Polynomial, Eulerian Graph, Cycle Graph
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1 Introduction

The definition of interlace polynomials was originated when there was a need to count the number of 2-in, 2-out digraphs having a given number of Euler circuits in an Eulerian graph raised from DNA sequencing by hybridization. Research has shown that special values of the interlace polynomial of a graph G can provide information about some structural properties of G. Interlace polynomials share similar properties as Martin Polynomials and Kauffman polynomials, which encode information about the families of closed paths in Eulerian graphs [4]. In this paper, we investigate a special type of Eulerian graph that is built from a cycle by adding a triangle to each edge of the cycle. We develop formulas for the interlace polynomials of such graphs, find properties of such polynomials, and apply them to describe some structural properties of the ground graphs.

Consider a simple graph $G=(V(G),E(G))$. For a vertex $v\in V(G)$, $N(v)$ denotes the set of neighbors of v, that is, $N(v)=\{\text{all vertices of G } adjacent \text{ to } v\}$. The resulting graph by removing the vertex v from G and all the edges adjacent to v is denoted $G-v$. The calculation of the interlace polynomial of a graph G starts from building the pivot of G. Consider an undirected non-empty graph G and an edge $ab\in E(G)$ with $a,b\in V(G)$. The edge ab determines three neighboring classes: (1) the vertices adjacent to both a and b, (2) the vertices adjacent to a alone excluding b, and (3) the vertices adjacent to b alone but not including a. In [4], a toggling process is applied to construct the pivot of a graph.

Definition 1.1 *Let* $G=(V(G),E(G))$ *be any undirected non-empty simple graph,* $a,b\in V(G)$*, and* $ab\in E(G)$*. We first partition the neighbors of* a *and* b *into three classes:*

1. $N(a)\setminus(\{b\}\cup N(b))$,
2. $N(b)\setminus(\{a\}\cup N(a))$,
3. $N(a)\cap N(b)$.

The pivot graph $G^{ab}=(V(G^{ab}),E(G^{ab}))$ of G, with respect to the edge ab, is the resulting graph with the same vertex set: $V(G)=V(G^{ab})$. The edge set is given by the toggling process: $\forall u,v\in V(G)$ with u, v belonging to two different classes of (1), (2), (3) shown above, $uv\in E(G)\Leftrightarrow uv\notin E(G^{ab})$ (Refer to Figure 1.)

Note that $G^{ab}=G^{ba}$. The process of obtaining the pivot graph G^{ab} from a graph G on an edge ab of G is called the pivot operation (or the toggling process). It is specifically defined on an edge of G. The definition for the interlace polynomial of a simple graph G involves a toggling process and is defined recursively as follows.

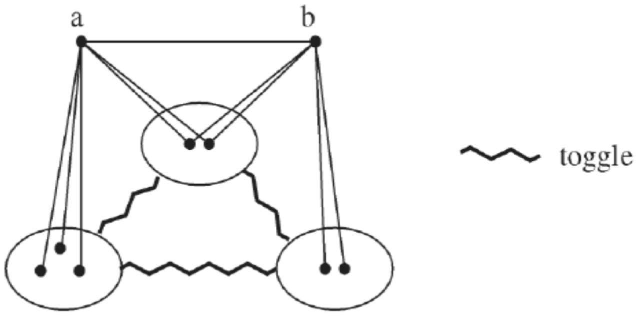

Figure 1: [4] The Pivot Operation on the Edge ab

Definition 1.2 *([4] Interlace Polynomial) Let G be any undirected simple graph with n vertices ($n > 0$). The interlace polynomial $q(G, x)$ of G is given below:*

1. If G is empty (no edge), $q(G, x) = x^n$.
2. If G is connected and has at least one edge $ab \in E(G)$, where $a, b \in V(G)$, then
$$q(G, x) = q(G - a, x) + q(G^{ab} - b, x).$$
3. If $G = G_1 \cdots G_k$ is the disjoint union of k connected simple graphs $G_1, \ldots, G_k$, then $q(G, x) = q(G_1, x)q(G_2, x) \cdots q(G_k, x).$

By Theorem 12 in [4], the map q is well defined on all simple graphs, that is, the polynomial $q(G, x)$ is independent on the selection of the edge ab. Below we give some known results that relate the interlace polynomials to the structural components of the ground graphs.

Theorem 1.3 *Let G be any simple graph. The following results hold:*

1. The degree of the lowest-degree term of $q(G, x)$ is $\kappa(G)$, the number of disconnected components of G;
2. $\deg(q(G, x)) \geq \alpha(G)$, where $\alpha(G)$ is the independence number, that is, the size of a maximal independent vertex set of G;
3. If G is a forest with n vertices, then $\deg(q(G, x)) = n - \mu(G)$, where $\mu(G)$ denotes the size of a maximum matching in G.

Explicit or recursive formulas for some special graphs such as paths, cycles, stars, and complete graphs can be found in literature. We summarize them below.

Lemma 1.4 *Let m, n be positive integers. The interlace polynomials are known for the following graphs [3]:*

1. (complete graph K_n with n vertices) $q(K_n, x) = 2^{n-1}x$;
2. (complete bipartite graph $K_{m,n}$)
$q(K_{m,n}, x) = (1 + x + \cdots + x^{m-1})(1 + x + \cdots + x^{n-1}) + x^m + x^n - 1$;
3. (path P_n with n edges) $q(P_1, x) = 2x$, $q(P_2, x) = x^2 + 2x$, and for $n \geq 3$, $q(P_n, x) = q(P_{n-1}, x) + xq(P_{n-2}, x)$;
4. (cycles) $q(C_3, x) = 4x$, $q(C_4, x) = 3x^2 + 2x$, and for $n > 4$, $q(C_n, x) = q(C_{n-2}, x) + q(P_{n-2}, x) + xq(P_{n-4}, x)$.
5. (star S_n with n edges, $n \geq 2$) $q(S_n, x) = x^n + x^{n-1} + \cdots + x^2 + 2x.$

We are interested in a type of Eulerian graph, denoted by Γ_n. For $n \geq 3$, the graph Γ_n is derived from the cycle C_n where each edge of C_n is used to build an additional 3-cycle (triangle) C_3. In Section 2, we give the definition of Γ_n and introduce three related graphs Δ_n, Λ_n, and W_n. To develop recursive and explicit formulas for the interlace polynomial of Γ_n, we perform the pivot operation on Γ_n in a certain way so that the resulting graphs involve the graphs Δ_n, Λ_n, and W_n, and other simple graphs whose interlace polynomials are already known. In section 3, we develop explicit formulas for the interlace polynomials of Δ_n, Λ_n, and W_n. In Section 4, we develop recursive and explicit formulas for the interlace polynomial of Γ_n. Properties of $q(\Gamma_n, x)$ are given in Section 5, which include patterns of the coefficients and some special values of the polynomial $q(\Gamma_n, x)$. Lastly, in Section 6, we give an application of the interlace polynomial $q(\Gamma_n, x)$ in calculating a rank problem for a related matrix. Similarly, the interlace polynomials of Δ_n, Λ_n, and W_n are applied to calculate the ranks of 3 related matrices modulo 2.

2 The Graphs of Interest and Preliminary Results

We focus on a special type of Eulerian graph (a graph that contains an Eulerian circuit). The graph Γ_n ($n \geq 3$) is built from the cycle C_n (called the center cycle) by adding an additional cycle C_3 to each edge of the center cycle. By the definition, every vertex of Γ_n has degree two or four. We start by demonstrating the smallest such graph, Γ_3. We show the decomposition process and how the interlace polynomial is developed.

Example 2.1 *The graph Γ_3 is shown below, which is built by adding an additional triangle (C_3) to each edge of the center C_3. The graph has 6 vertices and 9 edges. The interlace polynomial of Γ_3 is* $q(\Gamma_3, x) = x^3 + 10x^2 + 8x.$ *We perform the toggling process on the edge ab.*

[scale=0.75] [fill] (0,0) circle [radius=0.075]; at (2.2,2.45) a ; (0,0)–(2,0); [fill] (2,0) circle [radius=0.075]; (0,0)–(1,1); at (0.75,1.1) c; [fill] (1,1) circle [radius=0.075]; (1,1)–(2,2); at (-0.3,0) e; [fill] (2,2) circle [radius=0.075]; at (2,-0.4) d ; at (4.3,0) f ; (2,0)–(4,0); [fill] (4,0) circle [radius=0.075]; (4,0)–(3,1); [fill] (3,1) circle [radius=0.075]; at (3.2,1.3) b ; (1,1)–(2,0); (2,0)–(3,1); (1,1)–(3,1); [thick, -, red] (3,1)–(2,2);

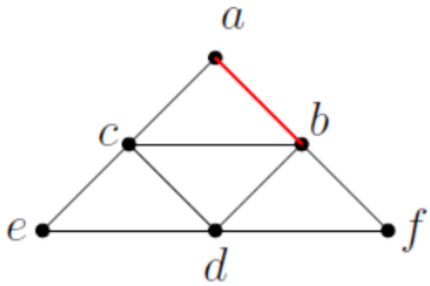

Figure 2: The Graph Γ_3 with the Selected Edge ab.

Note that the edge ab results in only two neighboring sets: $N(a) \cap N(b) = \{c\}$ and $N(b) \setminus (\{a\} \cup N(a)) = \{d, f\}$. The pivot Γ_3^{ab} has the same vertex set as that of Γ_3, but cf is added as an edge and cd is not an edge in the pivot. The graph Γ_3 and its pivot Γ_3^{ab} are shown below:

[scale=0.75] [fill] (0,0) circle [radius=0.075]; (1,1) circle [radius=0.45]; at (2.2,2.45) a; (0,0)–(2,0); [fill] (2,0) circle [radius=0.075]; (0,0)–(1,1); at (0.75,1.1) c; [fill] (1,1) circle [radius=0.075]; (1,1)–(2,2); at (-0.3,0) e; [fill] (2,2) circle [radius=0.075]; at (2,-0.4) d ; (3,0) circle [x radius=1.3, y radius=0.3]; at

(4.4,0) f ; (2,0)–(4,0); [fill] (4,0) circle [radius=0.075]; (4,0)–(3,1); [fill] (3,1) circle [radius=0.075]; at (3.2,1.3) b ; [thick, -, green] (1,1)–(2,0); (2,0)–(3,1); (1,1)–(3,1); (3,1)–(2,2); at (5.5,1) $\rightarrow$;

[fill] (7,0) circle [radius=0.075]; at (9.2,2.45) a ; (7,0)–(9,0); [fill] (9,0) circle [radius=0.075]; (7,0)–(8,1); at (7.75,1.1) c; [fill] (8,1) circle [radius=0.075]; (8,1)–(9,2); at (6.8,0) e; [fill] (9,2) circle [radius=0.075]; at (9,-0.4) d ; at (11.3,0) f ; (9,0)–(11,0); [fill] (11,0) circle [radius=0.075]; (11,0)–(10,1); [fill] (10,1) circle [radius=0.075]; at (10.2,1.3) b ; [thick, -, red] (8,1)–(11,0); (9,0)–(10,1); (8,1)–(10,1); (10,1)–(9,2); at (2, -1.2) Γ_3; at (9, -1.2) Γ_3^{ab} ;

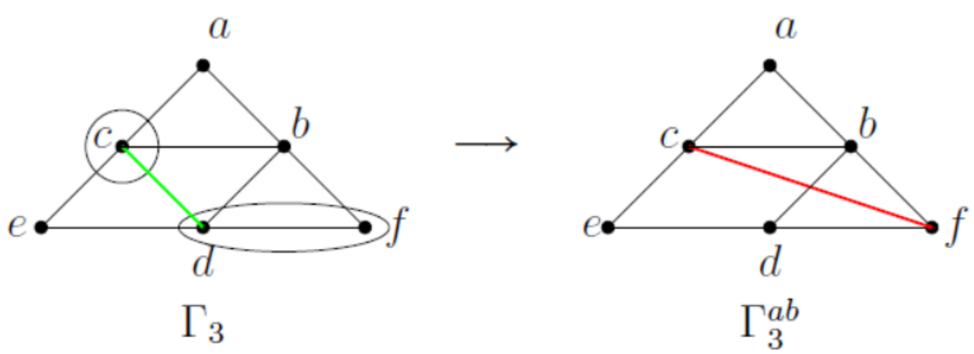

Figure 3: The Graph Γ_3 and its pivot Γ_3^{ab} .

We first “decompose" the graph Γ_3 into two smaller graphs $\Gamma_3 - a$ and $\Gamma_3^{ab} - b$:

[scale=0.75] [fill] (0,0) circle [radius=0.075]; (0,0)–(2,0); [fill] (2,0) circle [radius=0.075]; (0,0)–(1,1); at (0.75,1.1) c; [fill] (1,1) circle [radius=0.075]; (1,1)–(2,0); at (-0.3,0) e; at (2,-0.4) d ; at (4.4,0) f ; (2,0)–(4,0); [fill] (4,0) circle [radius=0.075]; (4,0)–(3,1); [fill] (3,1) circle [radius=0.075]; at (3.2,1.3) b ; [thick, -, green] (0,0)–(1,1); (2,0)–(3,1); (1,1)–(3,1); at (5.5,1) $+$;

[fill] (7,0) circle [radius=0.075]; at (9.2,2.45) a ; (7,0)–(9,0); [fill] (9,0) circle [radius=0.075]; (7,0)–(8,1); at (7.75,1.1) c; [fill] (8,1) circle [radius=0.075]; (8,1)–(9,2); at (6.8,0) e; [fill] (9,2) circle [radius=0.075]; at (9,-0.4) d ; at (11.3,0) f ; (9,0)–(11,0); [fill] (11,0) circle [radius=0.075]; (11,0)–(8,1); [->, red] (9,2)–(8,1); at (2, -1.2) $\Gamma_3 - a$; at (9, -1.2) $\Gamma_3^{ab} - b$;

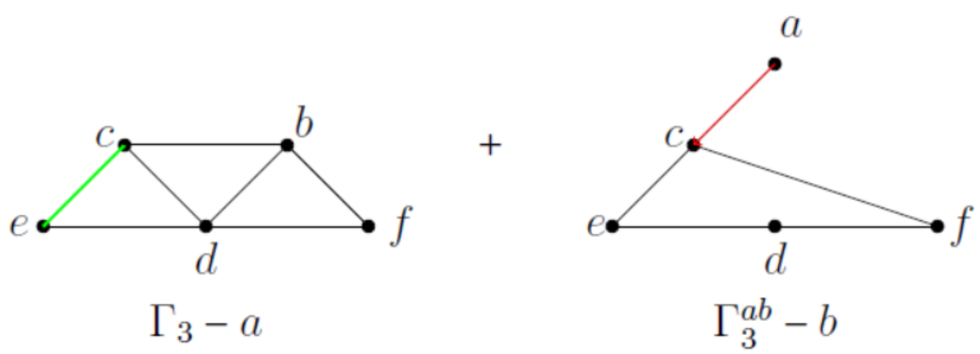

Figure 4: Decomposition by Toggling Γ_3 at ab.

Next, we toggle $\Gamma_3 - a$ at the edge ce:

[scale=0.75] [fill] (0,0) circle [radius=0.075]; (0,0)–(2,0); [fill] (2,0) circle [radius=0.075]; (0,0)–(1,1); at (0.75,1.1) c; [fill] (1,1) circle [radius=0.075]; at (-0.3,0) e; at (2,-0.4) d ; at (4.4,0) f ; (2,0)–(4,0); (2,0)–(1,1); [fill] (4,0) circle [radius=0.075]; (4,0)–(3,1); [fill] (3,1) circle [radius=0.075]; at (3.2,1.3) b ; [thick, -, green] (0,0)–(1,1); (2,0)–(3,1); (1,1)–(3,1); at (2, -1.5) $\Gamma_3^{ab} - a$; at (5.5,.5) $\rightarrow$; at (9, -1.5) Λ_1; at (14,-1.5) C_4; [fill] (7,0) circle [radius=0.075]; (7,0)–(9,0); [fill] (9,0) circle [radius=0.075]; (11,0)–(10,1);

at (6.75,0) e ; [fill] (10,1) circle [radius=0.075]; (9,0)–(11,0); at (9,-.4) d ; [fill] (11,0) circle [radius=0.075]; (9,0)–(10,1); at (12,.5) $+$;

(14,0)–(16,0); [fill] (14,0) circle [radius=0.075]; [fill] (13,1) circle [radius=0.075]; (13,1)–(14,0); (16,0)–(15,1); (13,1)–(15,1); [fill](16,0) circle [radius=0.075]; [fill](15,1) circle [radius=0.075];

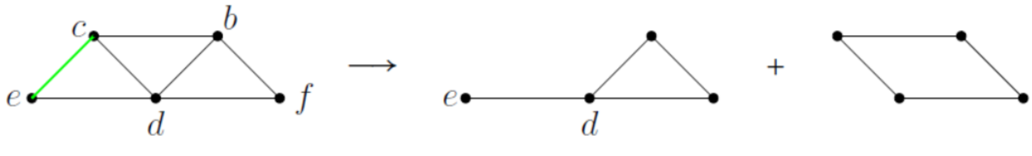

Figure 5: Pivoting $\Gamma_3 - a$ at the Edge ce

Furthermore we toggle $\Gamma_3^{ab} - b$ at the edge ac which results in C_4 and K_1P_2 (see the last piece in Figure 6). Note that the graph Λ_1 can be further decomposed into C_3 and K_1P_1 and thus the interlace polynomial of Γ_3 can be calculated through that of several smaller graphs derived from the above toggling process. These decomposition processes are shown below.

[scale=0.75] [fill] (0,0) circle [radius=0.075]; (0,0)–(2,0); [fill] (2,0) circle [radius=0.075]; (0,0)–(1,1); at (0.75,1.1) c ; [fill] (1,1) circle [radius=0.075]; [fill] (2,2) circle [radius=0.075]; at (2,-0.4) d ; at (4.4,0) f ; (2,0)–(4,0); [fill] (4,0) circle [radius=0.075]; (4,0)–(3,1); (1,1)–(2,2); (3,1)–(2,2); [fill] (3,1) circle [radius=0.075]; at (3.2,1.3) b ; (1,1)–(2,0); (2,0)–(3,1); (1,1)–(2,0); (1,1)–(3,1); at (5.5,1) $\rightarrow$;

[fill] (7,0) circle [radius=0.075]; (7,0)–(9,0); at (9,-.3) d ; at (7.7 ,1.1) c ; [fill] (7,0) circle [radius=0.075]; (7,0)–(8,1); (9,0)–(11,0); (9,0)–(8,1); at (3.2,1.3) b ; at (2,2.3) a ; at (-0.3,0) e ; [fill] (9,0) circle [radius=0.075]; [fill] (8,1) circle [radius=0.075]; at (6.7,0) e ; at (7.7,1.1) c ; at (11.3,0) f ; (9,0)–(11,0); [fill] (11,0) circle [radius=0.075]; (11,0)–(10,1); [fill] (10,1) circle [radius=0.075]; at (10.2,1.3) b ; (9,0)–(10,1); (8,1)–(10,1);

at (12,1) $+$;

at (12.7,0) e ; at (15,2.3) a ; at (17.3,0) f ; at (15,-.3) d ; (13,0)–(15,0); [fill] (13,0) circle [radius=0.075]; [fill] (15,0) circle [radius=0.075]; (15,0)–(17,0); (13,0)–(14,1); (14,1)–(15,2); [fill] (15,2) circle [radius=0.075]; [fill] (14,1) circle [radius=0.075]; [fill] (17,0) circle [radius=0.075]; (17,0)–(14,1); at (13.7,1.1) c ; at (2, -1.2) Γ_3 ; at (15, -1.2) $\Gamma_3^{ab} - b$; at (9, -1.2) $\Gamma_3 - a$;

at (0,-4) $\rightarrow$;

[fill] (1,-4.5) circle [radius=0.075]; (1,-4.5)–(3,-4.5); [fill] (3,-4.5) circle [radius=0.075]; (2,-3.5)–(1,-4.5); [fill] (2,-3.5) circle [radius=0.075]; (2,-3.5)–(3,-4.5);

at (4,-4) $+$;

[fill] (5,-4) circle [radius=0.075]; (6,-3.5)–(6,-4.5); [fill] (6,-3.5) circle [radius=0.075]; [fill] (6,-4.5) circle [radius=0.075];

at (7,-4) $+$;

(8,-3.5)–(8,-4.5); [fill] (8,-3.5) circle [radius=0.075]; [fill] (8,-4.5) circle [radius=0.075]; (9,-3.5)–(9,-4.5); (8,-3.5)–(9,-3.5); [fill] (9,-3.5) circle [radius=0.075]; [fill] (9,-4.5) circle [radius=0.075]; (8,-4.5)–(9,-4.5);

at (10,-4) $+$;

(11,-4)–(12,-4); [fill] (11,-4) circle [radius=0.075]; [fill] (12,-4) circle [radius=0.075]; (12,-4)–(13,-4); (11,-4)–(11.5,-3); [fill] (13,-4) circle [radius=0.075]; [fill] (11.5,-3) circle [radius=0.075]; (13,-4)–(11.5,-3); at (14,-4) $+$;

(15,-4)–(16,-4); [fill] (15,-4) circle [radius=0.075]; [fill] (16,-4) circle [radius=0.075]; (16,-4)–(17,-4); [fill] (17,-4) circle [radius=0.075]; [fill] (16,-3) circle [radius=0.075];

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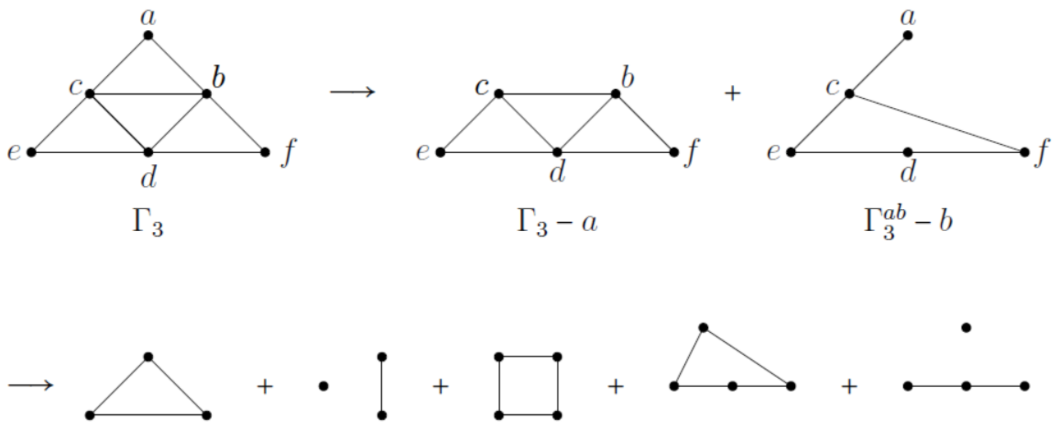

Figure 6: Decomposition of Γ_3 into Smaller Pieces.

Briefly, $\Gamma_3 \to (C_3 + K_1P_1 + C_4) + (C_4 + K_1P_2)$.

By Definition 1.2 and Lemma 1.4, we obtain

$$q(\Gamma_3, x) = q(C_3, x) + xq(P_1, x) + 2q(C_4, x) + xq(P_2, x)$$
$$= 4x + x(2x) + 2(3x^2 + 2x) + x(x^2 + 2x) = x^3 + 10x^2 + 8x.$$

Before we formally define Γ_n for $n > 3$, we introduce three related graphs Δ_n, Λ_n, and W_n for $n \geq 1$.

Definition 2.2 Let n be a positive integer.

1. The graph Δ_n is a "line-up" of n copies of C_3's shown below:

[scale=0.75] [fill] (0,0) circle [radius=0.075]; at (1,1.35) v_1 ; (0,0)–(2,0); (4,0)–(4.3,0); (6.7,0)–(7,0); [fill] (2,0) circle [radius=0.075]; (0,0)–(1,1); at (2,-0.35) u_2; [fill] (1,1) circle [radius=0.075]; at (3,1.35) v_2 ; at (8,1.35) v_{n-1} ; at (10,1.35) v_n ; at (4,-0.35) u_3 ; at (7,-0.35) u_{n-1} ; (2,0)–(4,0); (2,0)–(1,1); [fill] (4,0) circle [radius=0.075]; (4,0)–(3,1); [fill] (3,1) circle [radius=0.075]; at (9,-0.35) u_n ; (2,0)–(3,1); (3,0)–(4,0); at (5,0) $\cdots$; at (6,0) $\cdots$; [fill] (7,0) circle [radius=0.075]; (7,0)–(9,0); [fill] (9,0) circle [radius=0.075]; (9,0)–(10,1); at (0,-0.35) u_1 ; at (11,-.35) u_{n+1} ; (7,0)–(8,1); (9,0)–(11,0); [fill] (8,1) circle [radius=0.075]; [fill] (10,1) circle [radius=0.075]; (9,0)–(8,1); (11,0)–(10,1); [fill] (11,0) circle [radius=0.075];

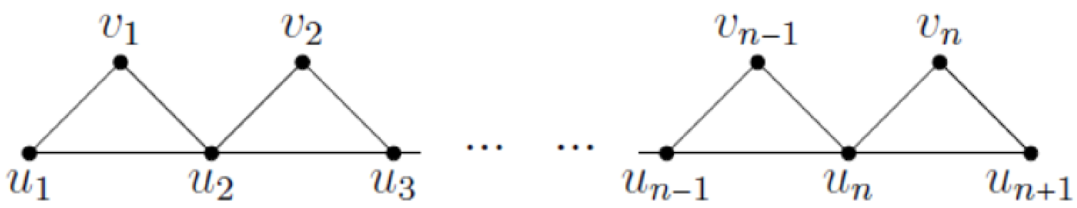

Figure 7: The Labeled Graph Δ_n.

2. Refer to Figure 2.2. The graph Λ_n is the resulting graph by adding a vertex u_0 and an edge u_0u_1 to Δ_n. Precisely, $\Lambda_n - u_0 = \Delta_n$. The graph of Λ_n is shown in the Figure below.

[scale=0.75] [fill] (0,0) circle [radius=0.075]; at (-0.6,0) u_0; at (2,-.35) u_1; at (3,1.35) v_1; at (10,1.40) v_n ; (0,0)–(2,0); [fill] (2,0) circle [radius=0.075]; (2,0)–(4,0); [fill] (4,0) circle [radius=0.075];

(4,0)–(3,1); at (11.35,-0.35) u_{n+1}; [fill] (3,1) circle [radius=0.075]; (2,0)–(3,1); (3,0)–(4,0); (4,0)–(4.3,0); (6.7,0)–(7,0); at (5,0) $\cdots$; at (6,0) $\cdots$; [fill] (7,0) circle [radius=0.075]; (7,0)–(9,0); [fill] (9,0) circle [radius=0.075]; (9,0)–(10,1); (7,0)–(8,1); (9,0)–(11,0); [fill] (8,1) circle [radius=0.075]; [fill] (10,1) circle [radius=0.075]; (9,0)–(8,1); (11,0)–(10,1); [fill] (11,0) circle [radius=0.075];

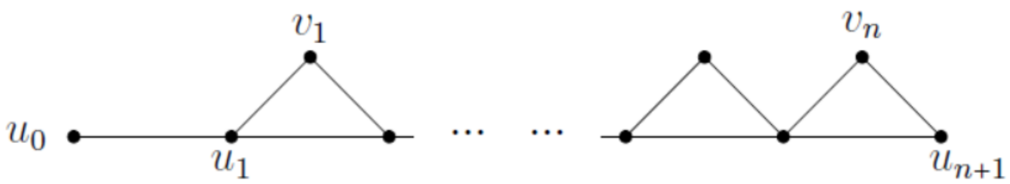

Figure 8: The Graph Λ_n Satisfying $\Lambda_n - u_0 \cong \Delta_n$.

3. Similarly, if we add one more vertex v_0 and one more edge $v_0 u_{n+1}$ at the other end of Λ_n, we obtain the graph W_n, which satisfies $W_n - u = \Lambda_n$ and $W_n - \{u_0, v_0\} = \Delta_n$. Here is the graph of W_n:

[scale=0.75] [fill] (0,0) circle [radius=0.075]; (0,0)–(2,0); [fill] (2,0) circle [radius=0.075]; at (-0.6,0) u_0; at (2,-.4) u_1; at (3,1.35) v_1; at (8,1.35) v_n; (2,0)–(4,0); [fill] (4,0) circle [radius=0.075]; (4,0)–(3,1); [fill] (3,1) circle [radius=0.075]; (2,0)–(3,1); (3,0)–(4,0); (4,0)–(4.3,0); (6.7,0)–(7,0); at (5,0) $\cdots$; at (6,0) $\cdots$; [fill] (7,0) circle [radius=0.075]; (7,0)–(9,0); [fill] (9,0) circle [radius=0.075]; (7,0)–(8,1); (9,0)–(11,0); at (9,-0.35) u_{n+1}; at (11.4,0) v_0; [fill] (8,1) circle [radius=0.075]; (9,0)–(8,1); [fill] (11,0) circle [radius=0.075];

Figure 9: The Graph W_n Satisfying $W_n - u_0 \cong \Lambda_n$.

First let us examine the case when $n = 3$. It is obvious that $\Delta_1 = C_3$. By toggling Λ_1 at the edge u_0u_1, Λ_1 is decomposed into $\Lambda_1 - u_0 = C_3$ and $\Lambda_1^{u_0u_1} - u_1 = K_1P_1$. Thus $q(\Lambda_3, x) = q(C_3, x) + xq(P_1, x)$. With a similar procedure, by toggling W_3 at the edge u_0u_1 in Figure 2.2, the graph W_1 decomposes into two graphs $W_1 - u_0 = \Lambda_1$ and $W_1^{u_0u_1} - u_1 = K_1P_2$. By Lemma 1.4, $q(C_3, x) = 4x, q(P_1, x) = 2x$, and $q(P_2, x) = x^2 + 2x$. Thus the interlace polynomials of Δ_1, Λ_1, and W_1 are obtained as follows.

Lemma 2.3

1. $q(\Delta_1, x) = q(C_3, x) = 4x$;
2. $q(\Lambda_1, x) = 2x(x+2)$;
3. $q(W_1, x) = x(x+2)^2$.

Now we define the main graph of interest in this study, named as Γ_n.

Definition 2.4 *For $n \geq 3$, the graph Γ_n is the resulting graph by gluing the two end vertices, b and c, of Δ_n so that Γ_n has a center cycle C_n and a cycle C_3 (represented by a triangle) was build from each edge of* *the center* C_n.

For $n \geq 4$, the graph Γ_n with labeled vertices is shown below in Figure 10 below.

[scale=0.75] [fill] (0,0) circle [radius=0.075]; at (1,1.35) v_1; at (3,1.4) v_2; at (2,-0.3) u_2; at (-0.35, 0) u_1; at (4, -0.35) u_3; at (5.5, -1.85) u_n; (0,0)–(2,0); (4,0)–(4.3,0); (6.7,0)–(7,0); [fill] (2,0) circle [radius=0.075]; [fill] (9,-2) circle [radius=0.075]; [fill] (2,-2) circle [radius=0.075]; (0,0)–(1,1); (5.5,-1.5)–(9,-2); (5.5,-1.5)–(2,-2); (0,0)–(5.5,-1.5); (2,-2)–(0,0); [fill] (5.5,-1.5) circle [radius=0.075]; at (2,-2.4) v_n; at (9,-2.4) v_{n-1}; at (10,1.35) v_{n-2}; at (8,1.35) v_{n-3}; at (11.7,-0.1) u_{n-2}; [fill] (1,1) circle [radius=0.075]; at (5.5,-.75) C_n; (2,0)–(4,0); (2,0)–(1,1); [fill] (4,0) circle [radius=0.075]; (4,0)–(3,1); [fill] (3,1) circle [radius=0.075]; (2,0)–(3,1); (3,0)–(4,0); at (5,0) $\cdots$; at (6,0) $\cdots$; [fill] (7,0) circle [radius=0.075]; (7,0)–(9,0); [fill] (9,0) circle [radius=0.075]; (9,0)–(10,1); (7,0)–(8,1); (9,0)–(11,0); [fill] (8,1) circle [radius=0.075]; [fill] (10,1) circle [radius=0.075]; (9,0)–(8,1); (11,0)–(5.5,-1.5); (11,0)–(9,-2); (11,0)–(10,1); [fill] (11,0) circle [radius=0.075];

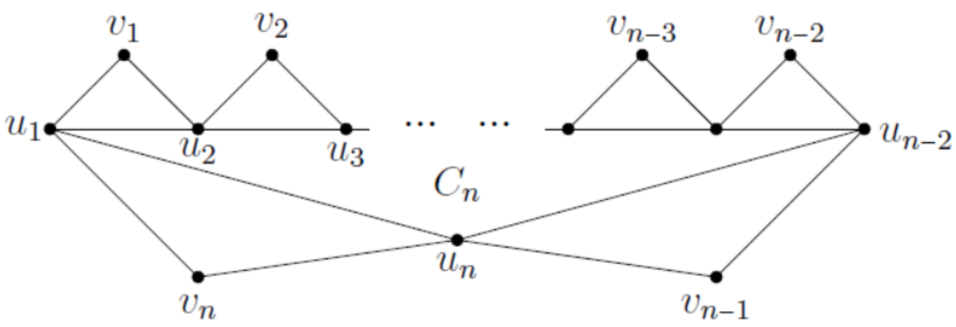

Figure 10: The Graph Γ_n with Labeled Vertices.

The graph Γ_n is made of n triangles (C_3) each sharing an edge with the center cycle C_n. In Figure 10, the top row has $n-2$ triangles. The main goal of this paper is to develop recursive and explicit formulas for the graph Γ_n. Later we show that a toggling process on Γ_n will result in 3 types of graphs: Δ_k, Λ_k, and W_k for some k with $1 \leq k \leq n-1$. We first study the interlace polynomials of these graphs.

The three graphs Δ_n, Λ_n, and W_n are closely related. As shown before, $\Lambda_n - u_0 \cong \Delta_n$, $W_n - u_0 \cong \Lambda_n$, $\Lambda_n - v_n \cong W_{n-1} \cong \Lambda_n^{v_n} u_{n+1} - u_{n+1}$, and $\Delta - \{u_0, v_0\} \cong W_{n-2}$. Below we give explicit formulas for the interlace polynomials of Δ_n, Λ_n, and W_n.

Theorem 2.5 *For* $n \geq 1$,

1. $q(\Lambda_n, x) = 2x(x+2)^n$;
2. $q(\Delta_n, x) = 2q(\Lambda_{n-1}, x) = 4x(x+2)^{n-1}$;
3. $q(W_n, x) = x(x+2)^{n+1}$.

Proof.

The results for $n = 1$ is shown in Lemma 2.3. For $n \geq 2$, the graph Δ_n has a simple recursive formula. Refer to Figure 2.2. By toggling the edge u_1v_1, we obtain two isomorphic graphs: $\Delta_n - u_1 \cong \Delta_n^{u_1v_1} - v_1 \cong \Lambda_{n-1}$. Thus $q(\Delta_n, x) = 2q(\Lambda_{n-1}, x)$. We then develop a recursive formula for $q(\Lambda_n, x)$ and use it to obtain explicit formulas for all the three graphs. By the toggling process on Λ_n at the edge u_0u_1 (refer to Figure 2.2), we obtain two graphs: $\Lambda_n - u_0 = \Delta_n$ and $\Lambda_n^{u_0u_1} - u_1 = K_1\Lambda_{n-1}$. Thus

$$q(\Lambda_n, x) = xq(\Lambda_{n-1}, x) + q(\Delta_n, x) = (x+2)q(\Lambda_{n-1}, x).$$

1. From Lemma 2.3, $q(\Lambda_1, x) = 2x(x+2)$. It is straightforward to check that

$$q(\Lambda_n, x) = (x+2)q(\Lambda_{n-1}, x) = (x+2)^2 q(\Lambda_{n-2}, x) = \cdots$$
$$= (x+2)^{n-1} q(\Lambda_1, x) = (x+2)^{n-1}(2x)(x+2) = 2x(x+2)^n.$$

2. From (1), $q(\Delta_n, x) = 2q(\Lambda_{n-1}, x) = 4x(x+2)^{n-1}$.
3. By toggling Λ_n at the edge $v_n u_{n+1}$, we have

$q(\Lambda_n, x) = q(\Lambda_n - v_n, x) + q(\Lambda_n^{v_n u_{n+1}} - u_{n+1}, x)$. But $\Lambda_n - v_n \cong \Lambda_n^{v_n u_{n+1}} - u_{n+1} \cong W_{n-1}$. Then $q(\Lambda_n, x) = 2q(W_{n-1}, x)$ and then $q(W_n, x) = q(\Lambda_{n+1}, x)/2 = x(x+2)^{n+1}$.

3 Properties of Γ_n and the Interlace Polynomial

By the definition of Γ_n, it is straightforward to prove the following basic graph theory properties of the graph Γ_n.

Theorem 3.1 *Refer to Figure 10 for the labeled graph Γ_n ($n \geq 3$).*

1. Γ_n is an Eulerian graph with $2n$ vertices and $3n$ edges. It has n vertices of degree 2 and n vertices of degree 4;
2. The independence number of Γ_n is n.
3. The size of a maximal matching of Γ_n is $\mu(G) = n$.
4. The edge-connectivity and vertex-connectivity of Γ_n are both 2.
5. The circumference of Γ_n is $|V(\Gamma_n)| = 2n$.
6. The diameter of Γ_n is $\frac{n+2}{2}$ if n is even and $\frac{n+1}{2}$ if n is odd.

Proof. (1) The degree of every vertex of Γ_n is even, so, Γ_n is Eulerian. For (2), a maximum independent set is given by all the n vertices of degree 2, that is, $\{v_1, v_2, \ldots, v_n\}$. (3) A maximal matching is made of the n edges $u_1v_1, u_2v_2, \ldots, u_nv_n$. (4) Since Γ_n is Eulerian, both edge-connectivity and vertex-connectivity are at least 2. The connectivity is 2 because Γ_n has a vertex of degree 2. (5) The Euler cycle $u_1v_1u_2v_2 \ldots u_nv_nu_1$ is the longest cycle. (6) When n is even, the distance between v_1 and $v_{(n+2)/2}$ is maximum by the path $v_1u_2u_3 \ldots u_{n/2}v_{n/2}u_{(n+2)/2}v_{(n+2)/2}$, which is of length $(n+2)/2$. So $d(v_1, v_{(n+2)/2}) = (n+2)/2$. If n is odd, the distance between v_1 and $v_{(n+1)/2}$ is maximal with $d(v_1, v_{(n+1)/2}) = (n+1)/2$. It is achieved by the path $v_1u_2u_3 \ldots u_{(n+1)/2}v_{(n+1)/2}$ which is of length $(n+1)/2$.

Next we develop a recursive and an explicit formula for the interlace polynomial of Γ_n.

Theorem 3.2 *Consider the graph Γ_n for $n \geq 3$.*

1. If $n > 3$, the interlace polynomial $q(\Gamma_n, x)$ satisfies the recursive relation:

$q(\Gamma_n, x) = 2q(\Gamma_{n-1}, x) + x(x+2)^{n-1}$.

2. Explicitly, for $n \geq 3$, $q(\Gamma_n, x) = 2^{n-1}(x^2 - x - 2) + (x+2)^n$.

Proof. (1) Refer to Figure 10. By applying the toggling process on Γ_n, with respect to the edge u_1v_1, we decompose Γ_n into two smaller graphs $\Gamma_n - v_1$ and $\Gamma_n^{u_1v_1} - u_1$. The graph $\Gamma_n^{u_1v_1}$ is the resulting graph by adding two edges u_2v_n and u_2u_n to Γ_n. The graph $\Gamma_n^{u_1v_1} - u_1 \cong \Gamma_{n-1}$ is shown below.

[scale=0.75] at (1,1.35) v_1; at (3,1.4) v_2; at (1.6,-0.28) u_2; at (4, -0.35) u_3; at (5.5, -1.85) u_n; (4,0)–(4.3,0); (6.7,0)–(7,0); [fill] (2,0) circle [radius=0.075]; [fill] (9,-2) circle [radius=0.075]; [fill] (2,-2) circle [radius=0.075]; (5.5,-1.5)–(9,-2); (5.5,-1.5)–(2,-2); (2,0)–(5.5,-1.5); (2,-2)–(2,0); [fill] (5.5,-1.5) circle [radius=0.075]; at (2,-2.4) v_n; at (9,-2.4) v_{n-1}; at (10,1.35) v_{n-2}; at (8,1.35) v_{n-3}; at (11.7,-0.1) u_{n-2}; [fill] (1,1) circle [radius=0.075]; at (5.5,-.75) C_{n-1}; (2,0)–(4,0); (2,0)–(1,1); [fill] (4,0) circle [radius=0.075]; (4,0)–(3,1); [fill] (3,1) circle [radius=0.075]; (2,0)–(3,1); (3,0)–(4,0); at (5,0) $\cdots$; at (6,0) $\cdots$; [fill] (7,0) circle [radius=0.075]; (7,0)–(9,0); [fill] (9,0) circle [radius=0.075]; (9,0)–(10,1); (7,0)–(8,1); (9,0)–(11,0); [fill] (8,1) circle [radius=0.075]; [fill] (10,1) circle [radius=0.075]; (9,0)–(8,1); (11,0)–(5.5,-1.5); (11,0)–(9,-2); (11,0)–(10,1); [fill] (11,0) circle [radius=0.075];

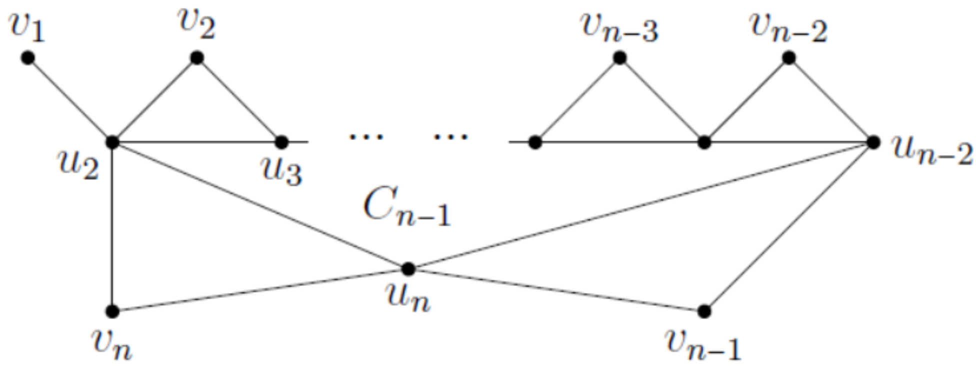

Figure 11: The Graph $\Gamma_n^{u_1v_1} - u_1$.

From the above decomposition, we have

$$q(\Gamma_n, x) = q(\Gamma_n - v_1, x) + q(\Gamma_n^{u_1v_1} - u_1, x).$$

Furthermore, we toggle the graph $\Gamma_n - v_1$ at the edge u_2v_2. Obviously, $(\Gamma_n - v_1) - u_2 \cong \Lambda_{n-2}$. The graph $(\Gamma_n - v_1)^{u_2v_2} - v_2 \cong \Gamma_{n-1}$ in the following way shown in the figure below (note the position change of u_2.) It gives $q(\Gamma_n - v_1, x) = q(\Lambda_{n-2}, x) + q(\Gamma_{n-1}, x)$.

[scale=0.75] [fill] (0,0) circle [radius=0.075]; at (1,1.35) u_2; at (3,1.4) v_3; at (2,-0.3) u_3; at (-0.35, 0) u_1; at (4, -0.35) u_4; at (5.5, -1.85) u_n; (0,0)–(2,0); (4,0)–(4.3,0); (6.7,0)–(7,0); [fill] (2,0) circle [radius=0.075]; [fill] (9,-2) circle [radius=0.075]; [fill] (2,-2) circle [radius=0.075]; (0,0)–(1,1); (5.5,-1.5)–(9,-2); (5.5,-1.5)–(2,-2); (0,0)–(5.5,-1.5); (2,-2)–(0,0); [fill] (5.5,-1.5) circle [radius=0.075]; at (2,-2.4) v_n; at (9,-2.4) v_{n-1}; at (10,1.35) v_{n-2}; at (8,1.35) v_{n-3}; at (11.7,-0.1) u_{n-2}; [fill] (1,1) circle [radius=0.075]; at (5.5,-.75) C_{n-1}; (2,0)–(4,0); (2,0)–(1,1); [fill] (4,0) circle [radius=0.075]; (4,0)–(3,1); [fill] (3,1) circle [radius=0.075]; (2,0)–(3,1); (3,0)–(4,0); at (5,0) $\cdots$; at (6,0) $\cdots$; [fill] (7,0) circle [radius=0.075]; (7,0)–(9,0); [fill] (9,0) circle [radius=0.075]; (9,0)–(10,1); (7,0)–(8,1); (9,0)–(11,0); [fill] (8,1) circle [radius=0.075]; [fill] (10,1) circle [radius=0.075]; (9,0)–(8,1); (11,0)–(5.5,-1.5); (11,0)–(9,-2); (11,0)–(10,1); [fill] (11,0) circle [radius=0.075];

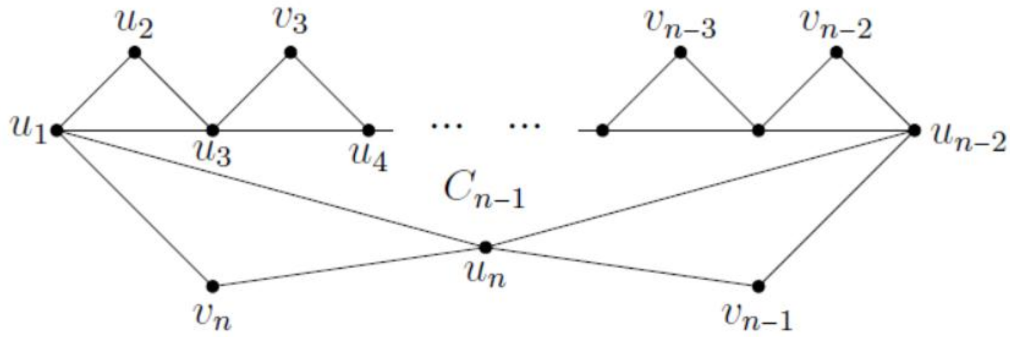

Figure 12: The Graph $(\Gamma_n - v_1)^{u_2 v_2} - v_2 \cong \Gamma_{n-1}$.

Next, we consider $\Gamma_n^{u_1 v_1} - u_1$ (see Figure 11) and perform the toggling process on it at the edge $v_1 u_2$. Removing v_1, it results in Γ_{n-1} again, that is, $\left(\Gamma_n^{u_1 v_1} - u_1\right) - v_1 \cong \Gamma_{n-1}$. The pivot $\left(\Gamma_n^{u_1 v_1} - u_1\right)^{v_1 u_2} \cong \Gamma_n^{u_1 v_1} - u_1$. One can easily check that $\left(\Gamma_n^{u_1 v_1} - u_1\right)^{v_1 u_2} - u_2 \cong K_1 W_{n-3}$. Thus $q(\Gamma_n^{u_1 v_1} - u_1, x) = q(\Gamma_{n-1}, x) + xq(W_{n-3})$.

Combining all the above together and applying Theorem 2.5, we obtain

$$\begin{aligned} q(\Gamma_n, x) &= 2q(\Gamma_{n-1}, x) + q(\Lambda_{n-2}, x) + xq(W_{n-3}, x) \\ &= 2q(\Gamma_{n-1}, x) + 2x(x+2)^{n-2} + x^2(x+2)^{n-2} \\ &= 2q(\Gamma_{n-1}, x) + x(x+2)^{n-1}. \end{aligned}$$

(2) For $n = 3$, $2^2(x^2 - x - 2) + (x+2)^3 = x^3 + 10x^2 + 8x$, which matches the formula for $q(\Gamma_3, x)$ given in Example 2.1. By mathematical induction, assume

$$q(\Gamma_{n-1}, x) = 2^{n-2}(x^2 - x - 2) + (x+2)^{n-1}.$$

Then by (1) and the induction hypothesis,

$$\begin{aligned} q(\Gamma_n, x) &= 2q(\Gamma_{n-1}, x) + x(x+2)^{n-1} \\ &= 2\left(2^{n-2}(x^2 - x - 2) + (x+2)^{n-1}\right) + x(x+2)^{n-1} \\ &= 2^{n-1}(x^2 - x - 2) + 2(x+2)^{n-1} + x(x+2)^{n-1} \\ &= 2^{n-1}(x^2 - x - 2) + (x+2)^n. \end{aligned}$$

Therefore, the explicit formula for $q(\Gamma_n, x)$ is proved.

Immediately from Theorem 3.2, we see that the polynomial $q(\Gamma_n, x)$ is of degree n and the leading coefficient is 1 for all $n \geq 3$. We are interested in finding other patterns and properties of the coefficients of the interlace polynomial of Γ_n. Below we list $q(\Gamma_n, x)$ for small values of n ranging from 3 to 8. These polynomials can be obtained by Theorem 3.2.

Example 3.3 *The interlace polynomials for* Γ_n*, with* $3 \leq n \leq 8$*, are as follows:*

1. $q(\Gamma_3, x) = x^3 + 10x^2 + 8x$;
2. $q(\Gamma_4, x) = x^4 + 8x^3 + 32x^2 + 24x$;
3. $q(\Gamma_5, x) = x^5 + 10x^4 + 40x^3 + 96x^2 + 64x$;

4. $q(\Gamma_6, x) = x^6 + 12x^5 + 60x^4 + 160x^3 + 272x^2 + 160x$;
5. $q(\Gamma_7, x) = x^7 + 14x^6 + 84x^5 + 280x^4 + 560x^3 + 736x^2 + 384x$.
6. $q(\Gamma_8, x) = x^8 + 16x^7 + 112x^6 + 448x^5 + 1120x^4 + 1792x^3 + 1920x^2 + 796x$.

A quick observation reveals that the second leading and the last coefficients of $q(\Gamma_n, x)$ seem to follow interesting patterns. Also, for each n, the coefficients show a “one mode" pattern. In the next section, we give some properties of the coefficients and special values of the polynomial.

4 Properties of $q(\Gamma_n, x)$

The interlace polynomial of a graph is a special graph invariant that can provide valuable different information about the graph. We are specifically interested in the coefficients and some special values of $q(\Gamma_n, x)$. Theorem 1.3 shows some examples that the coefficients and degree of an interlace polynomial reflect properties of the ground graph such as the connectivity, independence number, and the size of a maximum matching. It is also know that the value of the interlace polynomial $q(G, x)$ of a graph G at $x = -1$ can help in calculating the rank of a matrix related to the adjacency matrix of G modulo 2. In this section, we analyze some special values of the interlace polynomial $q(\Gamma_n, x)$ and identify patterns for the coefficients.

4.1 Coefficients of $q(\Gamma_n, x)$

From the explicit formulas given in Theorem 3.2, we can determine the coefficients of $q(\Gamma_n, x)$. Similarly, those of $q(\Lambda_n, x)$, $q(\Delta_n, x)$ and $q(W_n, x)$ can be obtained. First we focus on $q(\Gamma_n, x)$. Recall from Lemma 2.1 that $q(\Gamma_3, x) = x^3 + 10x^2 + 8x$.

Definition 4.1 *We define $a_{n,k}$ to be the coefficient of the x^k-term in the polynomial $q(\Gamma_n, x)$ ($k \geq 1$). That is,*

$$q(\Gamma_n, x) = \sum_{k=1}^{n} a_{n,k} x^k, \; n \geq 3.$$

Combining this definition and Theorem 3.2(2), we immediately derive the following

Theorem 4.2 *Consider the polynomial $q(\Gamma_n, x)$, where $n \geq 3$.*

1. The degree of $q(\Gamma_n, x)$ is n and the leading coefficient is $a_{n,n} = 1$.
2. The second leading coefficient is $a_{n,n-1} = 2n$.
3. The coefficients for the x-term and the x^2-term are $a_{n,1} = 2^{n-1}(n-1)$ and $a_{n,2} = 2^{n-3}(n^2 - n + 4)$ respectively.
4. If $2 < k < n-1$, then $a_{n,k} = 2^{n-k}\binom{n}{k}$.

Proof. The above result is true for $n = 3$ by Example 3.3. For $n \geq 4$, by Theorem 1.3, the constant term of the interlace polynomial of any connected graph is zero. It is so for $q(\Gamma_n, x)$. Applying the binomial expansion of $(x+2)^n$, we rewrite the explicit formula for $q(\Gamma_n, x)$ given in Theorem 3.2(2) as:

$$q(\Gamma_n, x) = 2^{n-1}x^2 - 2^{n-1}x + \sum_{k=1}^{n}\binom{n}{k}2^{n-k}x^k$$

$$= \sum_{k=3}^{n} \binom{n}{k} 2^{n-k} x^k + 2^{n-3}(n^2 - n + 4)x^2 + 2^{n-1}(n-1)x.$$

The statements are obvious then.

Example 3.3 lists $q(\Gamma_n, x)$ for $n = 3,4,5,6,7,8$. One can easily check that these polynomials confirms Theorem 4.2.

Another observation from Example 3.3 is that for every $n = 3,4,5,6,7,$ or 8, the sequence of coefficients $(a_{n,k})_{k=1}^{n}$ are one mode with the maximal value (peak) being $a_{n,2}$, the coefficient of the x^2-term. Is it true for every $n > 8$? We claim that

Proposition 4.3 *Let* $n \geq 3$ *and* $r_n = \lfloor \frac{n-1}{3} \rfloor$. *The sequence* $(a_{n,1})_{k=1}^{n}$ *is one mode: increasing-maximum-decreasing. Precisely,*

1. For $3 \leq n \leq 8$, the maximal value occurs at $k = 2$, that is, $a_{n,1} < a_{n,2} = \max$ and $a_{n,2} > a_{n,3} > \ldots > a_{n,n}$.
2. For $n \geq 9$ and $n \equiv 0$ or $1(\bmod 3)$, the maximal value occurs at $k = r_n = \lfloor \frac{n-1}{3} \rfloor$: $a_{n,1} < a_{n,2} < \ldots < a_{n,r_n} = \max > a_{n,r_n+1} \ldots > a_{n,n}$.
3. If $n \geq 9$ and $n \equiv 2(\bmod 3)$, then the maximal value occurs at both $k = r_n$ and $k = r_n + 1$. That is,

$$a_{n,1} < a_{n,2} < \ldots < a_{n,r_n} = \max = a_{n,r_n+1} > a_{n,r_n+1} \ldots > a_{n,n}.$$

Proof.

1. It is obvious by Lemma 3.3.
2. Refer to Equation ??. Assume $n \geq 9$ and $k \equiv 0$ or $1(\bmod 3)$. Then $r_n = \lfloor \frac{n-1}{3} \rfloor > \frac{n-2}{3}$. If $r_n \leq k \leq n-1$, then $3k - n + 2 > 0$ and so

$$a_{n,k} - a_{n,k+1} = 2^{n-k}\binom{n}{k} - 2^{n-(k+1)}\binom{n}{k+1} = \frac{n! 2^{n-k-1}(3k-n+2)}{k!(n-k)!} > 0.$$

Similarly, if $2 < k < \frac{n-2}{3}$, $a_{n,k} - a_{n,k+1} < 0$. From Theorem 4.2, $a_{n,n-1} = 2n > a_{n,n} = 1$ and $a_{n,1} = 2^{n-1}(n-1) < 2^{n-1}(n^2 - n + 4) = a_{n,2}$. For $n = 9$, $r_n = 2$ and the peak value is $a_{n,2} = 4864$. For $n > 9$, $r_n \geq 3$. We have $a_{n,1} < a_{n,2}$ and $a_{n,3} < a_{n,4} < \ldots < a_{n,r_n} > a_{n,r_n+1} \ldots > a_{n,n-1} = 2n > a_{n,n} = 1.$ It remains to show that $a_{n,2} < a_{n,3}$ for $n \geq 9$.

$$a_{n,3} - a_{n,2} = 2^{n-3}\binom{n}{3} - 2^{n-3}(n^2 - n + 4)$$

$$= \frac{2^{n-3}}{6}\left(n^2(n-9) + 8(n-3)\right) \geq \frac{2^{n-3}}{6}(48) = 2^n > 0.$$

Now we have shown that

$$a_{n,1} < a_{n,2} < \ldots < a_{n,r_n} = \max > a_{n,r_n+1} \ldots > a_{n,n-1} > a_{n,n}.$$

3. The proof of this part is similar as (2). When $n \equiv 2(\mathrm{mod}\,3)$, say, $n = 3m+2$, where m is a positive integer. Then $3m - n + 2 = 0$ implies $a_{n,m} = a_{n,m+1}$. The other inequalities are true as those in (2). So in this case the maximal value occurs at $k = m = r_n$ and $k = m+1 = r_n + 1$.

4.2 Special Values of $q(\Gamma_n, x)$

Research has shown that certain values of the interlace polynomial of a graph can provide useful information about the graph. Since all the coefficients are non-negative integers, the polynomial evaluated at any integer is also an integer. The following existing result describes the values of $q(G, x)$ at $x = 1, -1,$ and 2.

Theorem 4.4 *[1, 7] Let G be a graph with n vertices.*

1. $q(G,1)$ is the number of induced subgraphs of G with an odd number of perfect matchings (including the empty set).
2. $q(G,2) = 2^n$.
3. Let A be the ($n \times n$) adjacency matrix of G, and r be the rank of the matrix $A + I$ modulo 2, where I is the $n \times n$ identity matrix. Then $q(G,-1) = (-1)^r \cdot 2^{n-r} = (-1)^n (-2)^{n-r}$.

By Theorem 3.2, $q(\Gamma_n, 2) = (2+2)^n = 2^{2n}$ and $2n$ is the number of vertices of $q(\Gamma_n, x)$, which confirms Theorem 4.4(2). We evaluate $q(\Gamma_n, 1)$ and $q(\Gamma_n, -1)$ and then correlate the meaning to these results.

Proposition 4.5 For any positive integer $n \geq 3$,

1. The number of induced subgraphs of Γ_n with an odd number of perfect matchings is $3^n - 2^n$.
2. $q(\Gamma_n, -1) = 1$.
3. For any integer x, $q(\Gamma_n, x)$ has the same parity as that of x. That is, $q(\Gamma_n, x)$ is even if x is even and $q(\Gamma_n, x)$ is odd if x is odd.

Proof. By Theorem 3.2, $q(\Gamma_n, x) = 2^{n-1}(x^2 - x - 2) + (x+2)^n$. It gives $q(\Gamma_n, 1) = 3^n - 2^n$ and $q(\Gamma_n, -1) = 1$. By Theorem 4.4, the number of induced subgraphs of Γ_n with an odd number of perfect matchings is $3^n - 2^n$. Also, for $n \geq 3$, $2^{n-1}(x^2 - x - 2)$ is even. So, the parity of $q(\Gamma_n, x)$ is depending on that of $(x+2)^n$, and furthermore on the parity of x.

5 An Application in Matrix Theory

In this section, we use the interlace polynomial of a graph to calculate the rank of a related matrix modulo 2 as an application in linear algebra. In reference to Figure 10, we construct the adjacency matrix of Γ_n based on this order of the vertices: $v_1, u_2, v_2, u_3, \ldots, v_{n-1}, u_n, v_n, u_1$. Let us first look at the situation when $n = 5$.

Example 5.1 *Let A_{10} be the adjacency matrix of the graph Γ_5 and $B_{10} = A_{10} + I_{10}$, where I_{10} is the 10×10 identity matrix. The 10×10 matrix B_{10} is given below:*

$$B_{10} = A_{10} + I_{10} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

By calculating the determinant of B_{10} modulo 2, we obtain $|B_{10}| = 1 \neq 0$ in Z_2. Thus the rank of B_{10} is 10, that is, B_{10} is of full rank modulo 2.

Next we examine the structure of $B_{2n} = A_{2n} + I_{2n}$, where A_{2n} is the adjacency matrix of the graph Γ_n and I_{2n} is the $2n \times 2n$ identity matrix, for $n \geq 3$.

Lemma 5.2 *For any positive integer* $n \geq 3$,

$$A_{2n} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ddots & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 \end{bmatrix}_{2n \times 2n}$$

and

$$B_{2n} = A_{2n} + I = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ddots & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & \cdots & 1 & 1 & 1 \end{bmatrix}_{2n \times 2n}.$$

The structure of the matrix B_{2n} is described as follows. The first two rows and the last two rows are clearly shown above. Consider any integer k with $3 \leq k \leq 2n-2$. If k is odd, the k^{th} row is given by

$[0 \ \cdots \ 0 \ 1 \ 1 \ 1 \ 0 \ \cdots \ 0]$, where the first 1 of the 3 consecutive 1's occurs at the $(k-1)^{th}$ column. If k is even, then the k^{th} row is given by $[0 \ \cdots \ 0 \ 1 \ 1 \ 1 \ 1 \ 1 \ 0 \ \cdots \ 0]$, where the first 1 of the 5 consecutive 1's occurs at the $(k-2)^{th}$ column.

Using the interlace polynomial of Γ_n we can easily calculate the rank of B_{2n} (modulo 2) without performing any row or column reductions (the linear algebraic method).

Theorem 5.3 *Let A_{2n} be the adjacency matrix of Γ_n $(n \geq 3)$. The matrix $B_{2n} = I + A_{2n}$ is of full rank, that is, rank$(B_{2n}) = 2n$ modulo 2.*

Proof. Let $r =$ rank (B_{2n}) modulo 2. By Proposition 4.5(2), $q(\Gamma_n,-1) = 1$. Note that the graph Γ_n has $2n$ vertices. By Theorem 4.4(3),

$$(-1)^{2n}(-2)^{2n-r} = q(\Gamma_n,-1) = 1 \Rightarrow (-2)^{2n-r} = 1 \Rightarrow r = 2n.$$

Therefore B_{2n} is of full rank.

Of course the rank of matrix B_{2n} can be obtained by traditional linear algebra methods. One approach is described below. Refer to the structure of B_{2n} shown in Lemma 5.2. Perform the following elementary row or column operations:

1. For every m with $1 < m < n$, the $(2m)^{th}$ row subtracts the $(2m-1)^{th}$ row; The first three 1's are changed to three 0's. As a result, the last row has only two non-zero entries, both equal to 1, which occur at the first and second column;
2. Add row 1 to row 2. The entries at the $(2,1)$ and $(2,2)$ positions are changed to 0.
3. Add row 1 to the last row. The new last row now has only one non-zero entry valued 1 at the $(2n,2n)$ position;

The resulting matrix B_{2n}' after the above operations, which dose not change the rank of B_{2n}, has the following form:

$$B_{2n}' = \begin{bmatrix} F_1 & \mathbf{0} & \mathbf{0} & \cdots & \cdots & \mathbf{0} & 1 \\ \mathbf{0} & F_2 & * & \cdots & \cdots & * & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & F_2 & * & \cdots & * & \mathbf{0} \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & * & \vdots \\ \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0} & F_2 & \mathbf{0} \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{bmatrix},$$

where

$$F_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad and \quad F_2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

The matrix B_{2n}' is upper triangular and on the main diagonal there are $n-2$ F_2s. Obviously, modulo 2, the rank of F_1 is 3 and the rank of F_2 is 2. Thus the rank of B_{2n} is $3+2(n-2)+1 = 2n$. By comparison, using the interlace polynomial of Γ_n to evaluate this rank is quicker and easier.

Recall that from Theorem 2.5,

$$q(\Delta_n, x) = 4x(x+2)^{n-1},\ q(\Lambda_n, x) = 2x(x+2)^{n-1},\ q(W_n, x) = x(x+2)^{n-1}.$$

These polynomials can help us to find the ranks of some matrices related to the adjacency matrices of the 3 graphs: Δ_n, Λ_n, W_n. Refer to Figure 2.2 which shows the graph Δ_n with labeled vertices.

Let A_{Δ_n}, A_{Λ_n}, A_{W_n} be the adjacency matrix for Δ_n, Λ_n, W_n respectively. Let $B_{\Delta_n} = A_{\Delta_n} + I_{2n+1}$, $B_{\Lambda_n} = A_{\Lambda_n} + I_{2n+2}$, and $B_{W_n} = A_{W_n} + I_{2n+3}$. Similarly as the result in Theorem 5.3, we can evaluate the ranks of B_{Δ_n}, B_{Λ_n}, and B_{W_n} modulo 2.

Theorem 5.4 *Let r_{Δ_n}, r_{Λ_n}, and r_{W_n} be the rank of B_{Δ_n}, B_{Λ_n}, and B_{W_n} modulo 2 respectively. For any* $n \geq 2$,

$$r_{\Delta_n} = 2n-1, \quad r_{\Lambda_n} = 2n+1, \quad \textit{and} \quad r_{W_n} = 2n+3.$$

Proof. The numbers of vertices for B_{Δ_n}, B_{Λ_n}, and B_{W_n} are $2n+1$, $2n+2$, and $2n+3$ respectively. Theorem 2.5 shows that $q(\Delta_n, x) = 4x(x+2)^{n-1}$. By Theorem 4.4,

$$-4 = q(\Delta_n, -1) = (-1)^{2n+1} 2^{2n+1-r_{\Delta_n}} \Rightarrow 2^2 = 2^{2n+1-r_{\Delta_n}}$$
$$\Rightarrow 2 = 2n+1-r_{\Delta_n} \Rightarrow r_{\Delta_n} = 2n-1.$$

Similarly, $r_{\Lambda_n} = 2n+1$ and $r_{W_n} = 2n+3$.

The three square matrices have the following structures:

$$B_{\Delta_n} = \begin{bmatrix} 1 & 1 & 1 & & & & & & & & \\ 1 & 1 & 1 & & & & & & & & \\ 1 & 1 & 1 & 1 & 1 & & & & & & \\ & & 1 & 1 & 1 & & & & \mathbf{0} & & \\ & & 1 & 1 & 1 & 1 & 1 & & & & \\ & & & & & & \ddots & & & & \\ & & & & & & & \ddots & & & \\ & & & & \mathbf{0} & & & & & & \\ & & & & & & & 1 & 1 & 1 & 1 & 1 \\ & & & & & & & & & 1 & 1 & 1 \\ & & & & & & & & & 1 & 1 & 1 \end{bmatrix}_{2n+1}.$$

$$B_{\Lambda_n} = \begin{bmatrix} B_{\Delta_n} & F_3^T \\ F_3 & 1 \end{bmatrix}_{2n+2} \quad \textit{and} \quad B_{W_n} = \begin{bmatrix} 1 & F_4 & 0 \\ F_4^T & B_{\Delta_n} & F_3^T \\ 0 & F_3 & 1 \end{bmatrix}_{2n+3},$$

where $F_3 = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}_{1\times(2n+1)}$ and $F_4 = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}_{1\times(2n+1)}$. The ranks of these three matrices modulo 2 are obtained by using the interlace polynomials shown in the above theorem.

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