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A Study of the Page width of X-Trees in Book-Embedding

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Abstract: The book-embedding problem arises in connection with an approach to fault-tolerant VLSI design. An embedding of a simple undirected graph G in a book consists of an ordering of the vertices of G along the spine of the book, in which edges assigned to the same page do not cross. This study is aimed at X-tree graphs, an arbitrary X-tree graph has been known to guarantee two pages can be made on book-embeddings, and we provide some results for its page width of two pages.
Keywords: X-tree, book-embedding, page width.

1. Introduction

A book is a set of half-planes (the pages of the book) that share a common boundary line (the spine of the book). An embedding of a simple undirected graph of G (a pair of vertices are connected by at most one edge) in a book consists of an ordering of the vertices of G along the spine (horizontal line)of the book, together with an assignment of each edge of G to a page of thebook, in which edges assigned to the same page do not cross. This problem have application to several areas of theoretical computer science including VLSI design, algorithms, and complexity theory [1][3][4][6][7].

The minimum number of pages in which a graph G can be embedded is its page number, P(G). And the width of a page is the maximum number ofedges that cross any line perpendicular to the spine of the book. The widthof a book embedding, w(G), is the maximum width of any page of the book. The cumulative page width of a book embedding is the sum of the widths ofall the pages.

2. Main Result

The depth-d X-trees X(d) is the edge augmentation of the depth-d complete binary tree that adds edges going across each level of the tree in left-to-right order (see Figure 1).

X-trees are planar and subhamiltonian, hence admit 2-page embeddings. While it is easy to find a 2-page embedding for X(d) - the cycle that runs across levels in alternating orders yields one such - it is difficult to find one that has width o(n) (where $n = 2^d - 1$ is the number of vertices in X(d)), despite the fact that X(d) has a bisector of size d. However, the edge-augmentation of the X-tree depicted in Figure 1, with the indicated hamiltonian cycle, leads to the width-O(d) 2-page embedding of X(d) depicted in Figure 2.

Theorem 1. [2] The depth-d X-tree admits a 2-page embedding, with one page of width 2d and one of width 3d. This embedding is optimal in page number and is within a factor of five of optimal in cumulative page width.

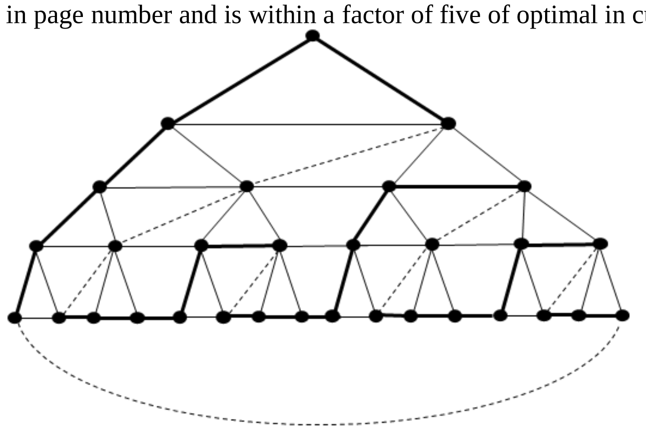

Figure 1: An edge-augmentation of the depth-5 X-tree and an efficient hamiltonian cycle which is given by Theorem1.

In the above known theorem, Chung, Leighton and Rosenberg gave anupper bound of the width of X(d). The Figure 1 and 2 are bang on indicating the efficient hamiltonian cycle which is used for embedding X(d)in the above theorem. For improving the bound, we give another 2-pagebook-embedding of X(d) and indicate the efficient hamiltonian cycle, see Figure 6 and 7.

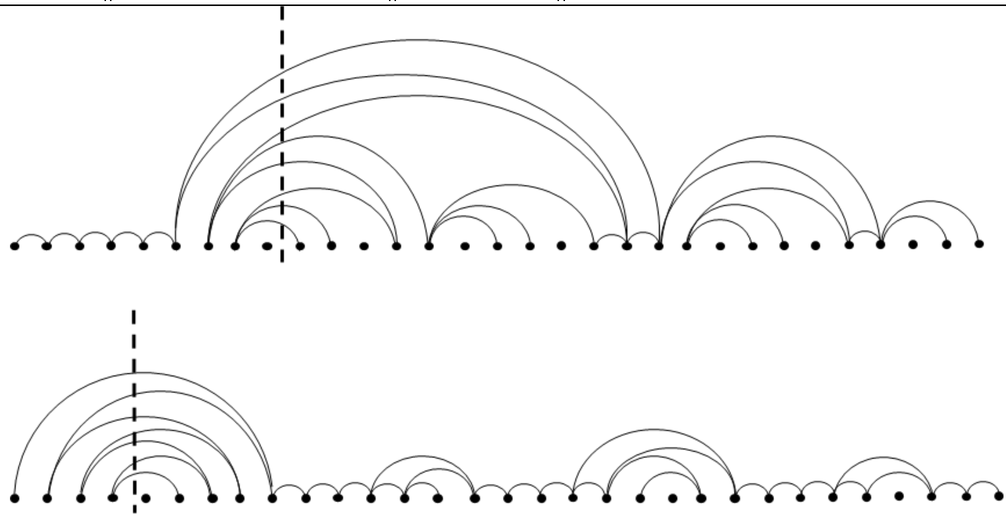

Figure 2: The embedding of the X-tree induced by the hamiltonian cycle of the Figure 1.

Theorem 2. The depth-d X-tree admits a 2-page embedding, withone page of width $\mathrm{d}-2$ and one of width $2\mathrm{d}-5$. This embedding is optimal in pagenumber and is within a factor of three of optimal in cumulative page width.

Proof. Optimality in number of pages is immediate since X(d) is not outer planar for $\mathrm{d} \geq 3$. The (near-) optimality of the claimed cutwidth follows from the proof in [5] that X(d) has no bisector of size less than d, coupledwith the demonstration that this implies a similar bound on cut width.

It remains only to verify that the widths of the pages in the prescribedembedding do indeed satisfy the claimed bounds. The verification proceedsby induction, but requires some detail about the layout of X(d). Say that wehave a 2-page embedding of $\mathrm{X}(\mathrm{d}-1)$ with the claimed pagewidths and thefollowing form. We depict the embedding schematically by its linearizationof X(d)'s vertices, together with a few relevant edges. For simplicity wedraw page 1 above the line of vertices and page 2 below the line. Since theembedding we construct is symmetric about the root, so it suffices to consider the left depth-$(\mathrm{d}-2)$ sub-X-tree.

Here r and s, are the root of $\mathrm{X}(\mathrm{d}-1)$ and its left son (i.e. the root of the left $\mathrm{X}(\mathrm{d}-2)$, respectively; α is the strings comprising the rest of $\mathrm{X}(\mathrm{d}-1)$'s vertices. Assume for induction that in the layout 1:

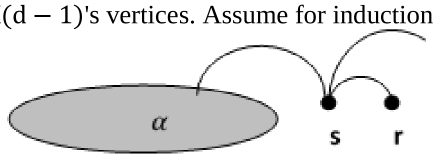

Figure 3: layout 1.

(1). The left spine vertices (which are the leftmost vertices at each level) of the left $\mathrm{X}(\mathrm{d}-2)$ appear consecutively in the left son of s-to-leaf in the leftmost of a (exactly the number of these vertices is $\mathrm{d}-3$);

(2). The right spine vertices of the left $\mathrm{X}(\mathrm{d}-2)$ appear consecutively in leaf-to-root in the rightmost of a (exactly the number of these vertices is $\mathrm{d}-3$);

(3). The vertices s and all of the left and right spine vertices are exposed on page 2, in the sence that no edge of the left $\mathrm{X}(\mathrm{d}-2)$ passes totally over them (i.e., under them in the picture);

(4). The width of page 1 in α is at most $2\mathrm{d}-7$.

(5). The width of page 2 in α is 1 in the consecutively left and right spine vertices areas, and less than $\mathrm{d}-5$ to the other non-spine vertices.

Now take a second reverse copy of α : α′ (see Figure 4).

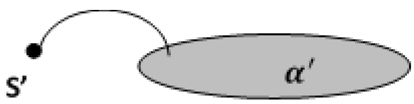

Figure 4: layout 2.

The prescribed layout of the left depth-(d − 1) sub-X-tree of X(d) – whose set of vertices is just the union of the sets of vertices of α , α′ , s , s′, in addition to s^*, its left son of the new root vertex r^* - is obtained from the indicated embedding as follows:

A careful analysis of the composite layout extends the induction: Conditions (1) and (2) are immediate since the left (resp., right) spine of the left sub-X(d − 1) is contained in the string s′α′ (resp, the string αs), whose order is inherited from layout 1 (resp, from layout 2). Condition (3) is clear from the depiction of layout 3: no edges are placed in the forbidden regions. Condition (4) and (5) are verified by simple counting.

Analysis of small X-trees establishes the base of the induction, completing the proof.

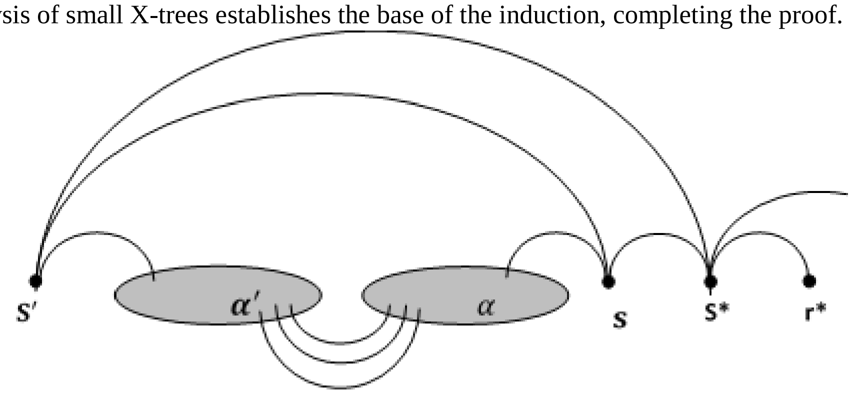

Figure 5: layout 3

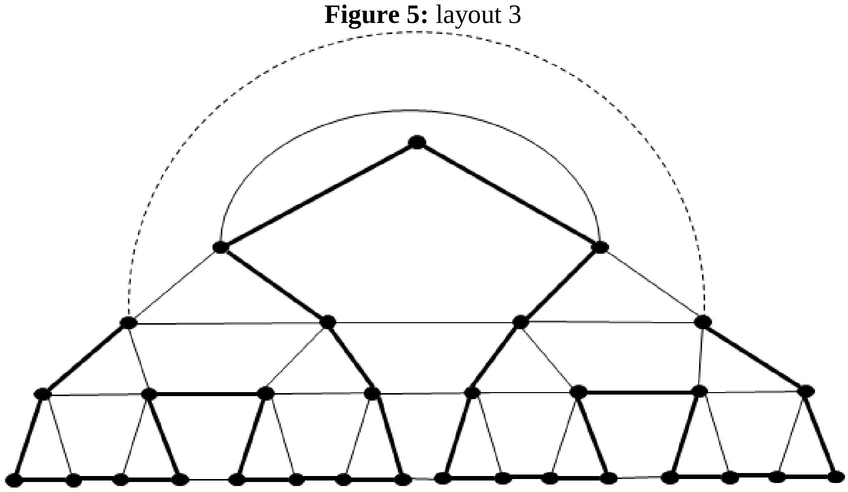

Figure 6: An edge-augmentation of the depth-5 X-tree and an efficient hamiltonian cycle which is given by Theorem2

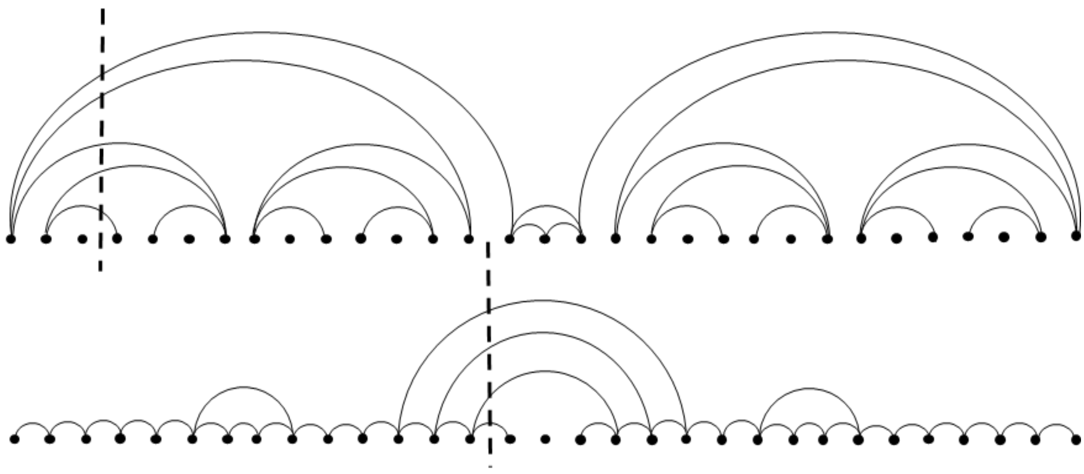

Figure 7: The embedding of the X-tree induced by the hamiltonian cycle of the Figure 6.

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