

## Positive $P_{0,1}$ Matrix Completion Problem for $5 \times 5$ Matrices Using Digraphs of Order 5 with 4 Arcs

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**Abstract:** A partial matrix is a matrix with some of its entries specified while others are free to be chosen. Completion of a partial matrix is a specific choice of unspecified entries such that the resulting matrix satisfies a certain property. This study considered positive  $P_{0,1}$  matrix completion problem which involved determining the patterns of positions with the property that any positive  $P_{0,1}$  that specified that pattern could be completed to a positive  $P_{0,1}$  matrix. Research on positive  $P_{0,1}$  has been done using digraphs of up to order 4. However, much has not been done using digraphs of order 5. This study therefore has focused on real  $5 \times 5$  partial positive  $P_{0,1}$  Matrices specifying digraphs for 5 vertices and 4 arcs. In particular, we determined all the digraphs of order 5 that led to completion, those that did not lead to completion and then classified the digraph characteristics that led to completion and the ones that led to non-completion. We constructed partial matrices from the digraphs using specified entries, unspecified and diagonal entries. The completion test was then performed on the partial matrices to determine the ones that lead to completion and the ones that lead to non-completion, as well as their classifications. We hope that the results of this study will advance further research on larger classes of matrices and richly add to the existing literature.

**Keywords:** Positive  $P_{0,1}$  Matrix completion, partial matrices, digraphs, principal minors, submatrices.

### 1. Introduction

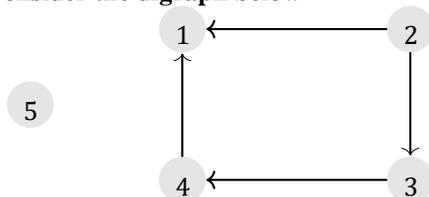
Matrix completion is a method of recovering lost information. It is achieved by filling in the missing entries of a partially observed matrix. A wide range of datasets are usually organized in matrix form. Situations may arise whereby a full set of data may not be available, or may be costly or not be economical to collect. Having knowledge that the complete data set should have some particular properties when arranged in matrix form, can aid in finding the values of the unavailable data. Hence the knowledge of matrix completion will be put into use in determining whether partially specified matrices can be completed to fully specified matrices satisfying certain prescribed properties. The knowledge and basic concept of graph theory will be useful in our study. Graph theory is the study of graphs, which are mathematical structures used to model pairwise relations between objects. A matrix  $A$  is a rectangular array or table of numbers, or symbols, arranged in rows and columns. A submatrix of a matrix  $A$  is a smaller matrix which is obtained by getting rid of some row(s) and/or column(s) from the matrix  $A$ . A real  $n \times n$  matrix is called a positive  $P_{0,1}$  matrix if all its entries are positive and all its principal minors are non-negative [6]. A positive partial  $P_{0,1}$  matrix is a partial matrix where all fully specified principal submatrices are positive  $P_{0,1}$  matrices [8]. If  $A$  is an  $n \times n$  matrix, for  $\beta$  subset of  $1, 2, \dots, n$ , the principal submatrix  $A(\beta)$  is obtained by getting rid of all the rows and columns that are not in  $\beta$ . A partial matrix is a matrix in which some entries are specified while others are not specified

and are free to be chosen [9] For example:  $A = \begin{bmatrix} 1 & 3 & 4 & -1 & a \\ 2 & 3 & -2 & b & 8 \\ 4 & 7 & 0 & 3 & -2 \\ c & 2 & 1 & 5 & 0 \\ -3 & 0 & d & 2 & 4 \end{bmatrix}$  is a

$5 \times 5$  partial matrix with elements in positions  $(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,5), (3,1), (3,2), (3,3), (3,4), (3,5), (4,2), (4,3), (4,4), (4,5), (5,1), (5,2), (5,4), (5,5)$  are specified whereas the elements in positions  $(1,5), (2,4), (4,1), (5,3)$  are not specified. A fully specified principal submatrix such as  $A(1,2,3)$  of matrix  $A$  above has all entries specified. Completion of a matrix is a specific choice of values for unspecified entries such that the resulting matrix specifies a certain property. Completion of a partial matrix is called zero completion if all the unspecified entries in the partial matrix are equated to zeros. The zero completion method involves setting all the unspecified entries to zero and then computing determinants of principal sub-matrices to figure out whether they are non-negative. If all the principal minors are non-negative the matrix is said to have completion. If otherwise it is said to be non-completable. For our case, since it involves positive  $P_{0,1}$ , where all entries have to be positive, zero completion method is not efficient. The area of study of matrix completion has attracted a lot of attention especially after the Netflix Problem, where matrices were used to rate movies where each customer is represented by row and the movie by column.

## 2. Acyclic Digraphs

### 1. Consider the digraph below



From the diagram above:

$$\text{Let } A = \begin{bmatrix} d_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ a_{21} & d_{22} & a_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & d_{33} & a_{34} & x_{35} \\ a_{41} & x_{42} & x_{43} & d_{44} & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{bmatrix} \text{ be the partial positive } P_0, 1$$

matrix specifying the digraph.

We have the determinants of the principal submatrices as:

$$\text{Det } A(1,2) = d_{11}d_{22} - x_{12}a_{21}$$

$$\text{Det } A(1,3) = d_{11}d_{33} - x_{13}x_{31}$$

$$\text{Det } A(1,4) = d_{11}d_{44} - x_{14}a_{41}$$

$$\text{Det } A(1,5) = d_{11}d_{55} - x_{15}x_{51}$$

$$\text{Det } A(2,3) = d_{22}d_{33} - a_{23}x_{32}$$

$$\text{Det } A(2,4) = d_{22}d_{44} - x_{24}x_{42}$$

$$\text{Det } A(2,5) = d_{22}d_{55} - x_{25}x_{52}$$

$$\text{Det } A(3,4) = d_{33}d_{44} - a_{34}x_{43}$$

$$\text{Det } A(3,5) = d_{33}d_{55} - x_{35}x_{53}$$

$$\text{Det } A(4,5) = d_{44}d_{55} - x_{45}x_{54}$$

$$\text{Det } A(1,2,3) = d_{11}(d_{22}d_{33} - a_{23}x_{32}) - x_{12}(a_{21}d_{33} - a_{23}x_{31}) + x_{13}(a_{21}x_{32} - d_{22}x_{31})$$

$$\text{Det } A(1,2,4) = d_{11}(d_{22}d_{44} - x_{24}x_{42}) - x_{12}(a_{21}d_{44} - x_{24}a_{41}) + x_{14}(a_{21}x_{42} - d_{22}a_{41})$$

$$\text{Det } A(1,2,5) = d_{11}(d_{22}d_{55} - x_{25}x_{52}) - x_{12}(a_{21}d_{55} - x_{25}x_{51}) + x_{15}(a_{21}x_{52} - d_{22}x_{51})$$

$$\text{Det } A(1,3,4) = d_{11}(d_{33}d_{44} - a_{34}x_{43}) - x_{13}(x_{31}d_{44} - a_{34}a_{41}) + x_{14}(x_{31}x_{43} - d_{33}a_{41})$$

$$\text{Det } A(1,3,5) = d_{11}(d_{33}d_{55} - x_{35}x_{53}) - x_{13}(x_{31}d_{55} - x_{35}x_{51}) + x_{15}(x_{31}x_{53} - d_{33}x_{51})$$

$$\text{Det } A(1,4,5) = d_{11}(d_{44}d_{55} - x_{45}x_{54}) - x_{14}(a_{41}d_{55} - x_{45}x_{51}) + x_{15}(a_{41}x_{54} - d_{44}x_{51})$$

$$\text{Det } A(2,3,4) = d_{22}(d_{33}d_{44} - a_{34}x_{43}) - a_{23}(x_{32}d_{44} - a_{34}x_{42}) + x_{24}(x_{32}x_{43} - d_{33}x_{42})$$

$$\text{Det } A(2,3,5) = d_{22}(d_{33}d_{55} - x_{35}x_{53}) - a_{23}(x_{32}d_{55} - x_{35}x_{52}) + x_{25}(x_{32}x_{53} - d_{33}x_{52})$$

$$\text{Det } A(2,4,5) = d_{22}(d_{44}d_{55} - x_{45}x_{54}) - x_{24}(x_{42}d_{55} - x_{45}x_{52}) + x_{25}(x_{42}x_{54} - d_{44}x_{52})$$

$$\text{Det } A(3,4,5) = d_{33}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})$$

$$\begin{aligned} \text{Det } A(1,2,3,4) = & d_{11}d_{22}(d_{33}d_{44} - a_{34}a_{43}) - a_{23}(x_{32}d_{44} - a_{34}x_{42}) + x_{24}(x_{32}x_{43} - d_{33}x_{42}) \\ & - x_{12}a_{21}(d_{33}d_{44} - a_{34}x_{43}) - a_{23}(x_{31}d_{44} - a_{34}a_{41}) + x_{24}(x_{31}x_{43} - d_{33}a_{41}) + x_{13}a_{21}(x_{32}d_{44} - \\ & x_{14}a_{21}(x_{32}x_{43} - d_{33}x_{42} - d_{22}(x_{31}x_{43} - d_{33}a_{41}) + a_{23}(x_{31}x_{42} - x_{32}a_{41})) \end{aligned}$$

$$\begin{aligned} \text{Det } A(1,2,3,5) = & d_{11}d_{22}(d_{33}d_{55} - x_{35}x_{53}) - a_{23}(x_{32}d_{55} - x_{35}x_{52}) + x_{25}(x_{32}x_{53} - d_{33}x_{52}) \\ & - x_{12}a_{21}(d_{33}d_{55} - x_{35}x_{53}) - a_{23}(x_{31}d_{55} - x_{35}x_{51}) + x_{25}(x_{31}x_{53} - d_{33}x_{51}) + x_{13}a_{21}(x_{32}d_{55} - \\ & x_{15}a_{21}(x_{32}x_{53} - d_{33}x_{52} - d_{22}(x_{31}x_{53} - d_{33}x_{51}) + a_{23}(x_{31}x_{52} - x_{32}x_{51})) \end{aligned}$$

$$\begin{aligned} \text{Det } A(1,2,4,5) = & d_{11}d_{22}(d_{44}d_{55} - x_{45}x_{54}) - x_{24}(x_{42}d_{55} - x_{45}x_{52}) + x_{25}(x_{42}x_{54} - d_{44}x_{52}) \\ & - x_{12}a_{21}(d_{44}d_{55} - x_{45}x_{54}) - x_{24}(a_{41}d_{55} - x_{45}x_{51}) + x_{25}(a_{41}x_{54} - d_{44}x_{51}) + x_{14}a_{21}(x_{42}d_{55} - \\ & x_{15}a_{21}(x_{42}x_{54} - d_{44}x_{52} - d_{22}(a_{41}x_{55} - d_{44}x_{51}) + x_{24}(a_{41}x_{52} - x_{42}x_{51})) \end{aligned}$$

$$\begin{aligned} \text{Det } A(1,3,4,5) = & d_{11}d_{33}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53}) \\ & - x_{13}x_{31}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(a_{41}d_{55} - x_{45}x_{51}) + x_{25}(a_{41}x_{54} - d_{44}x_{51}) + x_{14}x_{31}(x_{43}d_{55} - \\ & x_{15}x_{31}(x_{43}x_{54} - d_{44}x_{53} - d_{33}(a_{41}x_{54} - d_{44}x_{51}) + a_{34}(a_{41}x_{53} - x_{43}x_{51})) \end{aligned}$$

$$\begin{aligned} \text{Det } A(1,3,4,5) = & d_{11}d_{33}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53}) \\ & - x_{13}x_{31}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(a_{41}d_{55} - x_{45}x_{51}) + x_{25}(a_{41}x_{54} - d_{44}x_{51}) + x_{14}x_{31}(x_{43}d_{55} - \\ & x_{15}x_{31}(x_{43}x_{54} - d_{44}x_{53} - d_{33}(a_{41}x_{54} - d_{44}x_{51}) + a_{34}(a_{41}x_{53} - x_{43}x_{51})) \end{aligned}$$

$$\begin{aligned} \text{Det } A(2,3,4,5) = & d_{22}d_{33}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53}) \\ & - a_{23}x_{32}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{42}d_{55} - x_{45}x_{52}) + x_{25}(x_{42}x_{54} - d_{44}x_{52}) + x_{24}x_{32}(x_{43}d_{55} - \\ & x_{25}x_{32}(x_{43}x_{54} - d_{44}x_{53} - d_{33}(x_{42}x_{54} - d_{44}x_{52}) + a_{34}(x_{42}x_{53} - x_{43}x_{52})) \end{aligned}$$

$$\begin{aligned} \text{Det } A = & d_{11}[d_{22}(d_{33}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})) \\ & - a_{23}(x_{32}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{54} - d_{44}x_{52})) + \\ & x_{24}(x_{32}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{53} - x_{43}x_{52})) \\ & - x_{25}(x_{32}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{42}x_{54} - d_{44}x_{52}) + a_{34}(x_{42}x_{53} - x_{43}x_{52}))] - \end{aligned}$$

$$\begin{aligned} & x_{12}[a_{21}(d_{33}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})) - \\ & a_{23}(x_{31}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(a_{41}d_{55} - x_{45}x_{51}) + x_{35}(a_{41}x_{54} - d_{44}x_{51})) + \\ & x_{24}(x_{31}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(a_{41}d_{55} - x_{45}x_{51}) + x_{35}(a_{41}x_{53} - x_{43}x_{51})) - \\ & x_{25}(x_{31}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(a_{41}x_{54} - d_{44}x_{51}) + a_{34}(a_{41}x_{53} - x_{43}x_{51}))] + \end{aligned}$$

$$\begin{aligned} & x_{13}[a_{21}(x_{32}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{54} - d_{44}x_{52})) - \\ & d_{22}(x_{31}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(a_{41}d_{55} - x_{45}x_{51}) + x_{35}(a_{41}x_{54} - d_{44}x_{51})) + \\ & x_{24}(x_{31}(x_{42}d_{55} - x_{45}x_{52}) - x_{32}(a_{41}d_{55} - x_{45}x_{51}) + x_{35}(a_{41}x_{52} - x_{42}x_{51})) - \end{aligned}$$

$$\begin{aligned}
 & x_{25}(x_{31}(x_{42}x_{54}-d_{44}x_{52}) -x_{32}(a_{41}x_{54}-d_{44}x_{51}) + a_{34}(a_{41}x_{52}-x_{42}x_{51}))]- \\
 & x_{14}[a_{21}(x_{32}(x_{43}d_{55}-x_{45}x_{53})-d_{33}(x_{42}d_{55}-x_{45}x_{52})+x_{35}(x_{42}x_{53}-x_{43}x_{52}))]- \\
 & d_{22}(x_{31}(x_{43}d_{55} -x_{45}x_{53}) -d_{33}(a_{41}d_{55} -x_{45}x_{51}) + x_{35}(a_{41}x_{53} -x_{43}x_{51})) + \\
 & a_{23}(x_{31}(x_{42}d_{55} -x_{45}x_{52}) -x_{32}(a_{41}d_{55} -x_{45}x_{51}) + x_{35}(a_{41}x_{52} -x_{42}x_{51})) - \\
 & x_{25}(x_{31}(x_{42}x_{53}-x_{43}x_{52}) -x_{32}(a_{41}x_{53}-x_{43}x_{51}) + d_{33}(a_{41}x_{52}-x_{42}x_{51}))+ \\
 & x_{15}[a_{21}(x_{32}(x_{43}x_{54}-d_{44}x_{53})-d_{33}(x_{42}x_{54}-d_{44}x_{52})+a_{34}(x_{42}x_{53}-x_{43}x_{52}))]- \\
 & d_{22}(x_{31}(x_{43}d_{54} -d_{44}x_{53}) -d_{33}(a_{41}x_{54} -d_{44}x_{51}) + a_{34}(a_{41}x_{53} -x_{43}x_{51})) + \\
 & a_{23}(x_{31}(x_{42}x_{54} -d_{44}x_{52}) -x_{32}(a_{41}x_{54} -d_{44}x_{51}) + a_{34}(a_{41}x_{52} -x_{42}x_{51})) - \\
 & x_{24}(x_{31}(x_{42}x_{53}-x_{43}x_{52}) -x_{32}(a_{41}x_{53}-x_{43}x_{51}) + d_{33}(a_{41}x_{52}-x_{42}x_{51}))]
 \end{aligned}$$

By the definition of completion, the determinants should be greater or equals to zero. Therefore, we have:

$$d_{11}d_{22} - x_{12}a_{21} \geq 0 \Rightarrow d_{11}d_{22} = x_{12}a_{21} \Rightarrow x_{12} = \frac{d_{11}d_{22}}{a_{21}}$$

$$d_{11}d_{33} - x_{13}x_{31} \geq 0 \Rightarrow d_{11}d_{33} = x_{13}x_{31} \Rightarrow x_{13} = \frac{d_{11}d_{33}}{x_{31}}$$

$$d_{11}d_{44} - x_{14}a_{41} \geq 0 \Rightarrow d_{11}d_{44} = x_{14}a_{41} \Rightarrow x_{14} = \frac{d_{11}d_{44}}{a_{41}}$$

$$d_{11}d_{55} - x_{15}x_{51} \geq 0 \Rightarrow d_{11}d_{55} = x_{15}x_{51} \Rightarrow x_{15} = \frac{d_{11}d_{55}}{x_{51}}$$

$$d_{22}d_{33} - a_{23}x_{32} \geq 0 \Rightarrow d_{22}d_{33} = a_{23}x_{32} \Rightarrow x_{32} = \frac{d_{22}d_{33}}{a_{23}}$$

$$d_{22}d_{44} - x_{24}x_{42} \geq 0 \Rightarrow d_{22}d_{44} = x_{24}x_{42} \Rightarrow x_{24} = \frac{d_{22}d_{44}}{x_{42}}$$

$$d_{22}d_{55} - x_{25}x_{52} \geq 0 \Rightarrow d_{22}d_{55} = x_{25}x_{52} \Rightarrow x_{25} = \frac{d_{22}d_{55}}{x_{52}}$$

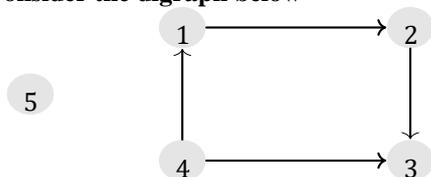
$$d_{33}d_{44} - a_{34}x_{43} \geq 0 \Rightarrow d_{33}d_{44} = a_{34}x_{43} \Rightarrow x_{43} = \frac{d_{33}d_{44}}{a_{34}}$$

$$d_{33}d_{55} - x_{35}x_{53} \geq 0 \Rightarrow d_{33}d_{55} = x_{35}x_{53} \Rightarrow x_{35} = \frac{d_{33}d_{55}}{x_{53}}$$

$$d_{44}d_{55} - x_{45}x_{54} \geq 0 \Rightarrow d_{44}d_{55} = x_{45}x_{54} \Rightarrow x_{45} = \frac{d_{44}d_{55}}{x_{54}}$$

Therefore substituting gives the partial matrix A a positive P 0,1 completion.

**2. Consider the digraph below**



From the digraph above:

$$\text{Let } A = \begin{bmatrix} d_{11} & a_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & d_{22} & a_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & d_{33} & x_{34} & x_{35} \\ a_{41} & x_{42} & a_{43} & d_{44} & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{bmatrix} \text{ be the partial positive } P_{0,1}$$

matrix specifying the digraph.

We have the determinants of the principal submatrices as:

$$\text{Det } A(1,2) = d_{11}d_{22} - a_{12}x_{21} \text{ Det}$$

$$A(1,3) = d_{11}d_{33} - x_{13}x_{31} \text{ Det}$$

$$A(1,4) = d_{11}d_{44} - x_{14}a_{41}$$

$$\text{Det } A(1,5) = d_{11}d_{55} - x_{15}x_{51} \text{ Det}$$

$$A(2,3) = d_{22}d_{33} - a_{23}x_{32} \text{ Det}$$

$$A(2,4) = d_{22}d_{44} - x_{24}x_{42}$$

$$\text{Det } A(2,5) = d_{22}d_{55} - x_{25}x_{52} \text{ Det}$$

$$A(3,4) = d_{33}d_{44} - x_{34}a_{43}$$

$$\text{Det } A(3,5) = d_{33}d_{55} - x_{35}x_{53}$$

$$\text{Det } A(4,5) = d_{44}d_{55} - x_{45}x_{54}$$

$$\text{Det } A(1,2,3) = d_{11}(d_{22}d_{33} - a_{23}x_{32}) - a_{12}(x_{21}d_{33} - a_{23}x_{31}) + x_{13}(x_{21}x_{32} - d_{22}x_{31})$$

$$\text{Det } A(1,2,4) = d_{11}(d_{22}d_{44} - x_{24}x_{42}) - a_{12}(x_{21}d_{44} - x_{24}a_{41}) + x_{14}(x_{21}x_{42} - d_{22}a_{41})$$

$$\text{Det } A(1,2,5) = d_{11}(d_{22}d_{55} - x_{25}x_{52}) - a_{12}(x_{21}d_{55} - x_{25}x_{51}) + x_{15}(x_{21}x_{52} - d_{22}x_{51})$$

$$\text{Det } A(1,3,4) = d_{11}(d_{33}d_{44} - x_{34}a_{43}) - x_{13}(x_{31}d_{44} - x_{34}a_{41}) + x_{14}(x_{31}a_{43} - d_{33}a_{41})$$

$$\text{Det } A(1,3,5) = d_{11}(d_{33}d_{55} - x_{35}x_{53}) - x_{13}(x_{31}d_{55} - x_{35}x_{51}) + x_{15}(x_{31}x_{53} - d_{33}x_{51})$$

$$\text{Det } A(1,4,5) = d_{11}(d_{44}d_{55} - x_{45}x_{54}) - x_{14}(a_{41}d_{55} - x_{45}x_{51}) + x_{15}(a_{41}x_{54} - d_{44}x_{51})$$

$$\text{Det } A(2,3,4) = d_{22}(d_{33}d_{44} - x_{34}a_{43}) - a_{23}(x_{32}d_{44} - x_{34}x_{42}) + x_{24}(x_{32}a_{43} - d_{33}x_{42})$$

$$\text{Det } A(2,3,5) = d_{22}(d_{33}d_{55} - x_{35}x_{53}) - a_{23}(x_{32}d_{55} - x_{35}x_{52}) + x_{25}(x_{32}x_{53} - d_{33}x_{52})$$

$$\text{Det } A(2,4,5) = d_{22}(d_{44}d_{55} - x_{45}x_{54}) - x_{24}(x_{42}d_{55} - x_{45}x_{52}) + x_{25}(x_{42}x_{54} - d_{44}x_{52})$$

$$\text{Det } A(3,4,5) = d_{33}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(a_{43}d_{55} - x_{45}x_{53}) + x_{35}(a_{43}x_{54} - d_{44}x_{53})$$

$$\text{Det } A(1,2,3,4) = d_{11}d_{22}(d_{33}d_{44}-x_{34}a_{43}) - a_{23}(x_{32}d_{44}-x_{34}x_{42}) + x_{24}(x_{32}a_{43}-d_{33}x_{42}) - a_{12}x_{21}(d_{33}d_{44}-x_{34}a_{43}) - a_{23}(x_{31}d_{44}-x_{34}a_{41}) + x_{24}(x_{31}a_{43}-d_{33}a_{41}) + x_{13}x_{21}(x_{32}d_{44}-x_{14}x_{21}(x_{32}a_{43}-d_{33}x_{42}-d_{22}(x_{31}a_{43}-d_{33}a_{41}) + a_{23}(x_{31}x_{42}-x_{32}a_{41}))$$

$$\text{Det } A(1,2,3,5) = d_{11}d_{22}(d_{33}d_{55}-x_{35}x_{53}) - a_{23}(x_{32}d_{55}-x_{35}x_{52}) + x_{25}(x_{32}x_{53}-d_{33}x_{52}) - a_{12}x_{21}(d_{33}d_{55}-x_{35}x_{53}) - a_{23}(x_{31}d_{55}-x_{35}x_{51}) + x_{25}(x_{31}x_{53}-d_{33}x_{51}) + x_{13}x_{21}(x_{32}d_{55}-x_{15}x_{21}(x_{32}x_{53}-d_{33}x_{52}-d_{22}(x_{31}x_{53}-d_{33}x_{51}) + a_{23}(x_{31}x_{52}-x_{32}x_{51}))$$

$$\text{Det } A(1,2,4,5) = d_{11}d_{22}(d_{44}d_{55}-x_{45}x_{54}) - x_{24}(x_{42}d_{55}-x_{45}x_{52}) + x_{25}(x_{42}x_{54}-d_{44}x_{52}) - a_{12}x_{21}(d_{44}d_{55}-x_{45}x_{54}) - x_{24}(a_{41}d_{55}-x_{45}x_{51}) + x_{25}(a_{41}x_{54}-d_{44}x_{51}) + x_{14}x_{21}(x_{42}d_{55}-x_{15}x_{21}(x_{42}x_{54}-d_{44}x_{52}-d_{22}(a_{41}x_{55}-d_{44}x_{51}) + x_{24}(a_{41}x_{52}-x_{42}x_{51}))$$

$$\text{Det } A(1,3,4,5) = d_{11}d_{33}(d_{44}d_{55}-x_{45}x_{54}) - x_{34}(a_{43}d_{55}-x_{45}x_{53}) + x_{35}(a_{43}x_{54}-d_{44}x_{53}) - x_{13}x_{31}(d_{44}d_{55}-x_{45}x_{54}) - x_{34}(a_{41}d_{55}-x_{45}x_{51}) + x_{25}(a_{41}x_{54}-d_{44}x_{51}) + x_{14}x_{31}(a_{43}d_{55}-x_{15}x_{31}(a_{43}x_{54}-d_{44}x_{53}-d_{33}(a_{41}x_{54}-d_{44}x_{51}) + x_{34}(a_{41}x_{53}-a_{43}x_{51}))$$

$$\text{Det } A(2,3,4,5) = d_{22}d_{33}(d_{44}d_{55}-x_{45}x_{54}) - x_{34}(a_{43}d_{55}-x_{45}x_{53}) + x_{35}(a_{43}x_{54}-d_{44}x_{53}) - a_{23}x_{32}(d_{44}d_{55}-x_{45}x_{54}) - x_{34}(x_{42}d_{55}-x_{45}x_{52}) + x_{25}(x_{42}x_{54}-d_{44}x_{52}) + x_{24}x_{32}(a_{43}d_{55}-x_{25}x_{32}(a_{43}x_{54}-d_{44}x_{53}-d_{33}(x_{42}x_{54}-d_{44}x_{52}) + x_{34}(x_{42}x_{53}-a_{43}x_{52}))$$

$$\text{Det } A = d_{11}[d_{22}(d_{33}(d_{44}d_{55}-x_{45}x_{54}) - x_{34}(a_{43}d_{55}-x_{45}x_{53}) + x_{35}(a_{43}x_{54}-d_{44}x_{53})) - a_{23}(x_{32}(d_{44}d_{55}-x_{45}x_{54}) - x_{34}(x_{42}d_{55}-x_{45}x_{52}) + x_{35}(x_{42}x_{54}-d_{44}x_{52})) + x_{24}(x_{32}(a_{43}d_{55}-x_{45}x_{53}) - d_{33}(x_{42}d_{55}-x_{45}x_{52}) + x_{35}(x_{42}x_{53}-a_{43}x_{52})) - x_{25}(x_{32}(a_{43}x_{54}-d_{44}x_{53}) - d_{33}(x_{42}x_{54}-d_{44}x_{52}) + x_{34}(x_{42}x_{53}-a_{43}x_{52}))]$$

$$- a_{12}[x_{21}(d_{33}(d_{44}d_{55}-x_{45}x_{54}) - x_{34}(a_{43}d_{55}-x_{45}x_{53}) + x_{35}(a_{43}x_{54}-d_{44}x_{53})) - a_{23}(x_{31}(d_{44}d_{55}-x_{45}x_{54}) - x_{34}(a_{41}d_{55}-x_{45}x_{51}) + x_{35}(a_{41}x_{54}-d_{44}x_{51})) + x_{24}(x_{31}(a_{43}d_{55}-x_{45}x_{53}) - d_{33}(a_{41}d_{55}-x_{45}x_{51}) + x_{35}(a_{41}x_{53}-a_{43}x_{51})) - x_{25}(x_{31}(a_{43}x_{54}-d_{44}x_{53}) - d_{33}(a_{41}x_{54}-d_{44}x_{51}) + x_{34}(a_{41}x_{53}-a_{43}x_{51}))]$$

$$+ x_{13}[x_{21}(x_{32}(d_{44}d_{55}-x_{45}x_{54}) - x_{34}(x_{42}d_{55}-x_{45}x_{52}) + x_{35}(x_{42}x_{54}-d_{44}x_{52})) - d_{22}(x_{31}(d_{44}d_{55}-x_{45}x_{54}) - x_{34}(x_{41}d_{55}-x_{45}x_{51}) + x_{35}(a_{41}x_{54}-d_{44}x_{51})) + x_{24}(x_{31}(x_{42}d_{55}-x_{45}x_{52}) - x_{32}(a_{41}d_{55}-x_{45}x_{51}) + x_{35}(a_{41}x_{52}-x_{42}x_{51})) - x_{25}(x_{31}(x_{42}x_{54}-d_{44}x_{52}) - x_{32}(a_{41}x_{54}-d_{44}x_{51}) + x_{34}(a_{41}x_{52}-x_{42}x_{51}))]$$

$$- x_{14}[x_{21}(x_{32}(a_{43}d_{55}-x_{45}x_{53}) - d_{33}(x_{42}d_{55}-x_{45}x_{52}) + x_{35}(x_{42}x_{53}-a_{43}x_{52})) -$$

$$\begin{aligned}
 & d_{22}(x_{31}(a_{43}d_{55} - x_{45}x_{53}) - d_{33}(a_{41}d_{55} - x_{45}x_{51}) + x_{35}(a_{41}x_{53} - a_{43}x_{51})) + \\
 & a_{23}(x_{31}(x_{42}d_{55} - x_{45}x_{52}) - x_{32}(a_{41}d_{55} - x_{45}x_{51}) + x_{35}(a_{41}x_{52} - x_{42}x_{51})) - \\
 & x_{25}(x_{31}(x_{42}x_{53} - a_{43}x_{52}) - x_{32}(a_{41}x_{53} - a_{43}x_{51}) + d_{33}(a_{41}x_{52} - x_{42}x_{51})) + \\
 & x_{15}[x_{21}(x_{32}(a_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{42}x_{54} - d_{44}x_{52}) + x_{34}(x_{42}x_{53} - a_{43}x_{52})) - \\
 & d_{22}(x_{31}(a_{43}d_{54} - d_{44}x_{53}) - d_{33}(a_{41}x_{54} - d_{44}x_{51}) + x_{34}(a_{41}x_{53} - a_{43}x_{51})) + \\
 & a_{23}(x_{31}(x_{42}x_{54} - d_{44}x_{52}) - x_{32}(a_{41}x_{54} - d_{44}x_{51}) + x_{34}(a_{41}x_{52} - x_{42}x_{51})) - \\
 & x_{24}(x_{31}(x_{42}x_{53} - a_{43}x_{52}) - x_{32}(a_{41}x_{53} - a_{43}x_{51}) + d_{33}(a_{41}x_{52} - x_{42}x_{51}))
 \end{aligned}$$

By the definition of completion, the determinants should be greater or equals to zero. Therefore, we have:

$$d_{11}d_{22} - a_{12}x_{21} \geq 0 \Rightarrow d_{11}d_{22} = a_{12}a_{21} \Rightarrow a_{12} = \frac{d_{11}d_{22}}{a_{21}}$$

$$d_{11}d_{33} - x_{13}x_{31} \geq 0 \Rightarrow d_{11}d_{33} = x_{13}x_{31} \Rightarrow x_{13} = \frac{d_{11}d_{33}}{x_{31}}$$

$$d_{11}d_{44} - x_{14}a_{41} \geq 0 \Rightarrow d_{11}d_{44} = x_{14}a_{41} \Rightarrow x_{14} = \frac{d_{11}d_{44}}{a_{41}}$$

$$d_{11}d_{55} - x_{15}x_{51} \geq 0 \Rightarrow d_{11}d_{55} = x_{15}x_{51} \Rightarrow x_{15} = \frac{d_{11}d_{55}}{x_{51}}$$

$$d_{22}d_{33} - a_{23}x_{32} \geq 0 \Rightarrow d_{22}d_{33} = a_{23}x_{32} \Rightarrow x_{32} = \frac{d_{22}d_{33}}{a_{23}}$$

$$d_{22}d_{44} - x_{24}x_{42} \geq 0 \Rightarrow d_{22}d_{44} = x_{24}x_{42} \Rightarrow x_{24} = \frac{d_{22}d_{44}}{x_{42}}$$

$$d_{22}d_{55} - x_{25}x_{52} \geq 0 \Rightarrow d_{22}d_{55} = x_{25}x_{52} \Rightarrow x_{25} = \frac{d_{22}d_{55}}{x_{52}}$$

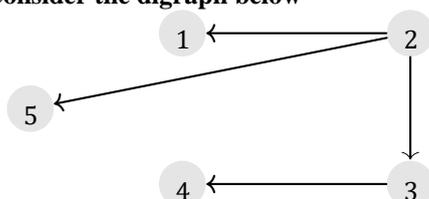
$$d_{33}d_{44} - x_{34}a_{43} \geq 0 \Rightarrow d_{33}d_{44} = x_{34}a_{43} \Rightarrow x_{34} = \frac{d_{33}d_{44}}{a_{43}}$$

$$d_{33}d_{55} - x_{35}x_{53} \geq 0 \Rightarrow d_{33}d_{55} = x_{35}x_{53} \Rightarrow x_{35} = \frac{d_{33}d_{55}}{x_{53}}$$

$$d_{44}d_{55} - x_{45}x_{54} \geq 0 \Rightarrow d_{44}d_{55} = x_{45}x_{54} \Rightarrow x_{45} = \frac{d_{44}d_{55}}{x_{54}}$$

Hence by substituting, we get a positive  $P_{0,1}$  completion.

**3. Consider the digraph below**



From the diagram above:

$$\text{Let } A = \begin{bmatrix} d_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ a_{21} & d_{22} & a_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & d_{33} & a_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & x_{45} \\ a_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{bmatrix} \text{ be the partial positive } P_{0,1}$$

matrix specifying the digraph

We have the determinants of the principal submatrices as:

$$\text{Det } A(1,2) = d_{11}d_{22} - x_{12}a_{21}$$

$$\text{Det } A(1,3) = d_{11}d_{33} - x_{13}x_{31}$$

$$\text{Det } A(1,4) = d_{11}d_{44} - x_{14}x_{41}$$

$$\text{Det } A(1,5) = d_{11}d_{55} - x_{15}x_{51}$$

$$\text{Det } A(2,3) = d_{22}d_{33} - a_{23}x_{32}$$

$$\text{Det } A(2,4) = d_{22}d_{44} - x_{24}x_{42}$$

$$\text{Det } A(2,5) = d_{22}d_{55} - x_{25}a_{52}$$

$$\text{Det } A(3,4) = d_{33}d_{44} - a_{34}x_{43}$$

$$\text{Det } A(3,5) = d_{33}d_{55} - x_{35}x_{53}$$

$$\text{Det } A(4,5) = d_{44}d_{55} - x_{45}x_{54}$$

$$\text{Det } A(1,2,3) = d_{11}(d_{22}d_{33} - a_{23}x_{32}) - x_{12}(a_{21}d_{33} - a_{23}x_{31}) + x_{13}(a_{21}x_{32} - d_{22}x_{31})$$

$$\text{Det } A(1,2,4) = d_{11}(d_{22}d_{44} - x_{24}x_{42}) - x_{12}(a_{21}d_{44} - x_{24}x_{41}) + x_{14}(a_{21}x_{42} - d_{22}x_{41})$$

$$\text{Det } A(1,2,5) = d_{11}(d_{22}d_{55} - x_{25}a_{52}) - x_{12}(a_{21}d_{55} - x_{25}x_{51}) + x_{15}(a_{21}a_{52} - d_{22}x_{51})$$

$$\text{Det } A(1,3,4) = d_{11}(d_{33}d_{44} - a_{34}x_{43}) - x_{13}(x_{31}d_{44} - a_{34}x_{41}) + x_{14}(x_{31}x_{43} - d_{33}x_{41})$$

$$\text{Det } A(1,3,5) = d_{11}(d_{33}d_{55} - x_{35}x_{53}) - x_{13}(x_{31}d_{55} - x_{35}x_{51}) + x_{15}(x_{31}x_{53} - d_{33}x_{51})$$

$$\text{Det } A(1,4,5) = d_{11}(d_{44}d_{55} - x_{45}x_{54}) - x_{14}(x_{41}d_{55} - x_{45}x_{51}) + x_{15}(x_{41}x_{54} - d_{44}x_{51})$$

$$\text{Det } A(2,3,4) = d_{22}(d_{33}d_{44} - a_{34}x_{43}) - a_{23}(x_{32}d_{44} - a_{34}x_{42}) + x_{24}(x_{32}x_{43} - d_{33}x_{42})$$

$$\text{Det } A(2,3,5) = d_{22}(d_{33}d_{55} - x_{35}x_{53}) - a_{23}(x_{32}d_{55} - x_{35}a_{52}) + x_{25}(x_{32}x_{53} - d_{33}a_{52})$$

$$\text{Det } A(2,4,5) = d_{22}(d_{44}d_{55} - x_{45}x_{54}) - x_{24}(x_{42}d_{55} - x_{45}a_{52}) + x_{25}(x_{42}x_{54} - d_{44}a_{52})$$

$$\text{Det } A(3,4,5) = d_{33}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})$$

$$\text{Det } A(1,2,3,4) = d_{11}d_{22}(d_{33}d_{44} - a_{34}a_{43}) - a_{23}(x_{32}d_{44} - a_{34}x_{42}) + x_{24}(x_{32}x_{43} - d_{33}x_{42}) - x_{12}a_{21}(d_{33}d_{44} - a_{34}x_{43}) - a_{23}(x_{31}d_{44} - a_{34}x_{41}) + x_{24}(x_{31}x_{43} - d_{33}x_{41}) + x_{13}a_{21}(x_{32}d_{44} - x_{14}a_{21}(x_{32}x_{43} - d_{33}x_{42} - d_{22}(x_{31}x_{43} - d_{33}x_{41}) + a_{23}(x_{31}x_{42} - x_{32}x_{41}))$$

$$\text{Det A}(1,2,3,5) = d_{11}d_{22}(d_{33}d_{55}-x_{35}x_{53}) - a_{23}(x_{32}d_{55}-x_{35}a_{52}) + x_{25}(x_{32}x_{53}-d_{33}d_{55}) - x_{12}a_{21}(d_{33}d_{55}-x_{35}x_{53}) - a_{23}(x_{31}d_{55}-x_{35}x_{51}) + x_{25}(x_{31}x_{53}-d_{33}x_{51}) + x_{13}a_{21}(x_{32}d_{55} - x_{15}a_{21}(x_{32}x_{53}-d_{33}a_{52}-d_{22}(x_{31}x_{53}-d_{33}x_{51}) + a_{23}(x_{31}a_{52}-x_{32}x_{51}))$$

$$\text{Det A}(1,2,4,5) = d_{11}d_{22}(d_{44}d_{55}-x_{45}x_{54}) - x_{24}(x_{42}d_{55}-x_{45}a_{52}) + x_{25}(x_{42}x_{54}-d_{44}d_{55}) - x_{12}a_{21}(d_{44}d_{55}-x_{45}x_{54}) - x_{24}(x_{41}d_{55}-x_{45}x_{51}) + x_{25}(x_{41}x_{54}-d_{44}x_{51}) + x_{14}a_{21}(x_{42}d_{55} - x_{15}a_{21}(x_{42}x_{54}-d_{44}a_{52}-d_{22}(x_{41}x_{55}x_{54}-d_{44}x_{51}) + x_{24}(x_{41}a_{52}-x_{42}x_{51}))$$

$$\text{Det A}(1,3,4,5) = d_{11}d_{33}(d_{44}d_{55}-x_{45}x_{54}) - x_{34}(x_{43}d_{55}-x_{45}x_{53}) + x_{35}(x_{43}x_{54}-d_{44}d_{55}) - x_{13}x_{31}(d_{44}d_{55}-x_{45}x_{54}) - a_{34}(x_{41}d_{55}-x_{45}x_{51}) + x_{25}(x_{41}x_{54}-d_{44}x_{51}) + x_{14}x_{31}(x_{43}d_{55} - x_{15}x_{31}(x_{43}x_{54}-d_{44}x_{53}-d_{33}(x_{41}x_{54}-d_{44}x_{51}) + a_{34}(x_{41}x_{53}-x_{43}x_{51}))$$

$$\text{Det A}(2,3,4,5) = d_{22}d_{33}(d_{44}d_{55}-x_{45}x_{54}) - a_{34}(x_{43}d_{55}-x_{45}x_{53}) + x_{35}(x_{43}x_{54}-d_{44}d_{55}) - a_{23}x_{32}(d_{44}d_{55}-x_{45}x_{54}) - a_{34}(x_{42}d_{55}-x_{45}a_{52}) + x_{25}(x_{42}x_{54}-d_{44}a_{52}) + x_{24}x_{32}(x_{43}d_{55} - x_{25}x_{32}(x_{43}x_{54}-d_{44}x_{53}-d_{33}(x_{42}x_{54}-d_{44}a_{52}) + a_{34}(x_{42}x_{53}-x_{43}a_{52}))$$

$$\text{Det A} = d_{11}[d_{22}(d_{33}(d_{44}d_{55}-x_{45}x_{54}) - a_{34}(x_{43}d_{55}-x_{45}x_{53}) + x_{35}(x_{43}x_{54}-d_{44}x_{53})) - a_{23}(x_{32}(d_{44}d_{55}-x_{45}x_{54}) - a_{34}(x_{42}d_{55}-x_{45}a_{52}) + x_{35}(x_{42}x_{54}-d_{44}a_{52})) + x_{24}(x_{32}(x_{43}d_{55}-x_{45}x_{53}) - d_{33}(x_{42}d_{55}-x_{45}a_{52}) + x_{35}(x_{42}x_{53}-x_{43}a_{52})) - x_{25}(x_{32}(x_{43}x_{54}-d_{44}x_{53}) - d_{33}(x_{42}x_{54}-d_{44}a_{52}) + a_{34}(x_{42}x_{53}-x_{43}a_{52}))]$$

$$x_{12}[a_{21}(d_{33}(d_{44}d_{55}-x_{45}x_{54}) - a_{34}(x_{43}d_{55}-x_{45}x_{53}) + x_{35}(x_{43}x_{54}-d_{44}x_{53})) - a_{23}(x_{31}(d_{44}d_{55}-x_{45}x_{54}) - a_{34}(x_{41}d_{55}-x_{45}x_{51}) + x_{35}(x_{41}x_{54}-d_{44}x_{51})) + x_{24}(x_{31}(x_{43}d_{55}-x_{45}x_{53}) - d_{33}(x_{41}d_{55}-x_{45}x_{51}) + x_{35}(x_{41}x_{53}-x_{43}x_{51})) - x_{25}(x_{31}(x_{43}x_{54}-d_{44}x_{53}) - d_{33}(x_{41}x_{54}-d_{44}x_{51}) + a_{34}(x_{41}x_{53}-x_{43}x_{51}))]$$

$$x_{13}[a_{21}(x_{32}(d_{44}d_{55}-x_{45}x_{54}) - a_{34}(x_{42}d_{55}-x_{45}a_{52}) + x_{35}(x_{42}x_{54}-d_{44}a_{52})) - d_{22}(x_{31}(d_{44}d_{55}-x_{45}x_{54}) - a_{34}(x_{41}d_{55}-x_{45}x_{51}) + x_{35}(x_{41}x_{54}-d_{44}x_{51})) + x_{24}(x_{31}(x_{42}d_{55}-x_{45}a_{52}) - x_{32}(x_{41}d_{55}-x_{45}x_{51}) + x_{35}(x_{41}a_{52}-x_{42}x_{51})) - x_{25}(x_{31}(x_{42}x_{54}-d_{44}a_{52}) - x_{32}(x_{41}x_{54}-d_{44}x_{51}) + a_{34}(x_{41}a_{52}-x_{42}x_{51}))]$$

$$x_{14}[a_{21}(x_{32}(x_{43}d_{55}-x_{45}x_{53}) - d_{33}(x_{42}d_{55}-x_{45}x_{52}) + x_{35}(x_{42}x_{53}-x_{43}a_{52})) - d_{22}(x_{31}(x_{43}d_{55}-x_{45}x_{53}) - d_{33}(x_{41}d_{55}-x_{45}x_{51}) + x_{35}(x_{41}x_{53}-x_{43}x_{51})) + a_{23}(x_{31}(x_{42}d_{55}-x_{45}a_{52}) - x_{32}(x_{41}d_{55}-x_{45}x_{51}) + x_{35}(x_{41}a_{52}-x_{42}x_{51})) - x_{25}(x_{31}(x_{42}x_{53}-x_{43}a_{52}) - x_{32}(x_{41}x_{53}-x_{43}x_{51}) + d_{33}(x_{41}a_{52}-x_{42}x_{51}))]$$

$$x_{15}[a_{21}(x_{32}(x_{43}x_{54}-d_{44}x_{53}) - d_{33}(x_{42}x_{54}-d_{44}a_{52}) + a_{34}(x_{42}x_{53}-x_{43}a_{52})) - d_{22}(x_{31}(x_{43}d_{54}-d_{44}x_{53}) - d_{33}(x_{41}x_{54}-d_{44}x_{51}) + a_{34}(x_{41}x_{53}-x_{43}x_{51})) + a_{23}(x_{31}(x_{42}x_{54}-d_{44}a_{52}) - x_{32}(x_{41}x_{54}-d_{44}x_{51}) + a_{34}(x_{41}a_{52}-x_{42}x_{51})) - x_{24}(x_{31}(x_{42}x_{53}-x_{43}a_{52}) - x_{32}(x_{41}x_{53}-x_{43}x_{51}) + d_{33}(x_{41}a_{52}-x_{42}x_{51}))]$$

By the definition of completion, the determinants should be greater or equals to zero. Therefore, we have:

$$d_{11}d_{22} - x_{12}a_{21} \geq 0 \Rightarrow d_{11}d_{22} = x_{12}a_{21} \Rightarrow x_{12} = \frac{d_{11}d_{22}}{a_{21}}$$

$$d_{11}d_{33} - x_{13}x_{31} \geq 0 \Rightarrow d_{11}d_{33} = x_{13}x_{31} \Rightarrow x_{13} = \frac{d_{11}d_{33}}{x_{31}}$$

$$d_{11}d_{44} - x_{14}x_{41} \geq 0 \Rightarrow d_{11}d_{44} = x_{14}x_{41} \Rightarrow x_{14} = \frac{d_{11}d_{44}}{x_{41}} \quad d_{11}d_{55} - x_{15}x_{51} \geq$$

$$0 \Rightarrow d_{11}d_{55} = x_{15}x_{51} \Rightarrow x_{15} = \frac{d_{11}d_{55}}{x_{51}} \quad d_{22}d_{33} - a_{23}x_{32} \geq 0 \Rightarrow d_{22}d_{33} =$$

$$a_{23}x_{32} \Rightarrow x_{32} = \frac{d_{22}d_{33}}{a_{23}} \quad d_{22}d_{44} - x_{24}x_{42} \geq 0 \Rightarrow d_{22}d_{44} = x_{24}x_{42} \Rightarrow x_{24} = \frac{d_{22}d_{44}}{x_{42}}$$

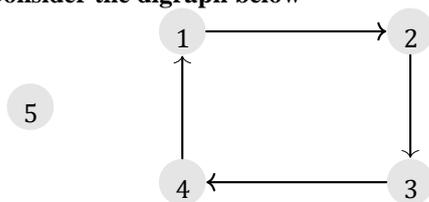
$$d_{22}d_{55} - x_{25}a_{52} \geq 0 \Rightarrow d_{22}d_{55} = x_{25}a_{52} \Rightarrow x_{25} = \frac{d_{22}d_{55}}{a_{52}} \quad d_{33}d_{44} - a_{34}x_{43} \geq$$

$$0 \Rightarrow d_{33}d_{44} = a_{34}x_{43} \Rightarrow x_{43} = \frac{d_{33}d_{44}}{a_{34}} \quad d_{33}d_{55} - x_{35}x_{53} \geq 0 \Rightarrow d_{33}d_{55} = x_{35}x_{53} \Rightarrow x_{35} =$$

$$\frac{d_{33}d_{55}}{x_{53}} \quad d_{44}d_{55} - x_{45}x_{54} \geq 0 \Rightarrow d_{44}d_{55} = x_{45}x_{54} \Rightarrow x_{45} = \frac{d_{44}d_{55}}{x_{54}}$$

### 3. Cyclic Digraphs

1. Consider the digraph below



From the diagram above:

Let  $A = \begin{bmatrix} d_{11} & a_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & d_{22} & a_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & d_{33} & a_{34} & x_{35} \\ a_{41} & x_{42} & x_{43} & d_{44} & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{bmatrix}$  be the partial positive  $P_{0,1}$

matrix specifying the digraph.

We have the determinants of the principal submatrices as:

$$\text{Det } A(1,2) = d_{11}d_{22} - a_{12}x_{21} \quad \text{Det}$$

$$A(1,3) = d_{11}d_{33} - x_{13}x_{31} \quad \text{Det}$$

$$A(1,4) = d_{11}d_{44} - x_{14}a_{41}$$

$$\text{Det } A(1,5) = d_{11}d_{55} - x_{15}x_{51} \quad \text{Det}$$

$$A(2,3) = d_{22}d_{33} - a_{23}x_{32} \quad \text{Det}$$

$$A(2,4) = d_{22}d_{44} - x_{24}x_{42}$$

$$\text{Det } A(2,5) = d_{22}d_{55} - x_{25}x_{52}$$

$$\text{Det } A(3,4) = d_{33}d_{44} - a_{34}x_{43} \quad \text{Det}$$

$$A(3,5) = d_{33}d_{55} - x_{35}x_{53}$$

$$\text{Det } A(4,5) = d_{44}d_{55} - x_{45}x_{54}$$

$$\text{Det } A(1,2,3) = d_{11}(d_{22}d_{33} - a_{23}x_{32}) - a_{12}(x_{21}d_{33} - a_{23}x_{31}) + x_{13}(x_{21}x_{32} - d_{22}x_{31})$$

$$\text{Det } A(1,2,4) = d_{11}(d_{22}d_{44} - x_{24}x_{42}) - a_{12}(x_{21}d_{44} - x_{24}a_{41}) + x_{14}(x_{21}x_{42} - d_{22}a_{41})$$

$$\text{Det } A(1,2,5) = d_{11}(d_{22}d_{55} - x_{25}x_{52}) - a_{12}(x_{21}d_{55} - x_{25}x_{51}) + x_{15}(x_{21}x_{52} - d_{22}x_{51})$$

$$\text{Det } A(1,3,4) = d_{11}(d_{33}d_{44} - a_{34}x_{43}) - x_{13}(x_{31}d_{44} - a_{34}a_{41}) + x_{14}(x_{31}x_{43} - d_{33}a_{41})$$

$$\text{Det } A(1,3,5) = d_{11}(d_{33}d_{55} - x_{35}x_{53}) - x_{13}(x_{31}d_{55} - x_{35}x_{51}) + x_{15}(x_{31}x_{53} - d_{33}x_{51})$$

$$\text{Det } A(1,4,5) = d_{11}(d_{44}d_{55} - x_{45}x_{54}) - x_{14}(a_{41}d_{55} - x_{45}x_{51}) + x_{15}(a_{41}x_{54} - d_{44}x_{51})$$

$$\text{Det } A(2,3,4) = d_{22}(d_{33}d_{44} - a_{34}x_{43}) - a_{23}(x_{32}d_{44} - a_{34}x_{42}) + x_{24}(x_{32}x_{43} - d_{33}x_{42})$$

$$\text{Det } A(2,3,5) = d_{22}(d_{33}d_{55} - x_{35}x_{53}) - a_{23}(x_{32}d_{55} - x_{35}x_{52}) + x_{25}(x_{32}x_{53} - d_{33}x_{52})$$

$$\text{Det } A(2,4,5) = d_{22}(d_{44}d_{55} - x_{45}x_{54}) - x_{24}(x_{42}d_{55} - x_{45}x_{52}) + x_{25}(x_{42}x_{54} - d_{44}x_{52})$$

$$\text{Det } A(3,4,5) = d_{33}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})$$

$$\begin{aligned} \text{Det } A(1,2,3,4) = & d_{11}d_{22}(d_{33}d_{44} - a_{34}x_{43}) - a_{23}(x_{32}d_{44} - a_{34}x_{42}) + x_{24}(x_{32}x_{43} - d_{33}x_{42}) \\ & - a_{12}x_{21}(d_{33}d_{44} - a_{34}x_{43}) - a_{23}(x_{31}d_{44} - a_{34}a_{41}) + x_{24}(x_{31}x_{43} - d_{33}a_{41}) + x_{13}x_{21}(x_{32}d_{44} - \\ & x_{14}x_{21}(x_{32}x_{43} - d_{33}x_{42} - d_{22}(x_{31}x_{43} - d_{33}a_{41}) + a_{23}(x_{31}x_{42} - x_{32}a_{41})) \end{aligned}$$

$$\begin{aligned} \text{Det } A(1,2,3,5) = & d_{11}d_{22}(d_{33}d_{55} - x_{35}x_{53}) - a_{23}(x_{32}d_{55} - x_{35}x_{52}) + x_{25}(x_{32}x_{53} - d_{33}x_{52}) \\ & - a_{12}x_{21}(d_{33}d_{55} - x_{35}x_{53}) - a_{23}(x_{31}d_{55} - x_{35}x_{51}) + x_{25}(x_{31}x_{53} - d_{33}x_{51}) + x_{13}x_{21}(x_{32}d_{55} - \\ & x_{15}x_{21}(x_{32}x_{53} - d_{33}x_{52} - d_{22}(x_{31}x_{53} - d_{33}x_{51}) + a_{23}(x_{31}x_{52} - x_{32}x_{51})) \end{aligned}$$

$$\begin{aligned} \text{Det } A(1,2,4,5) = & d_{11}d_{22}(d_{44}d_{55} - x_{45}x_{54}) - x_{24}(x_{42}d_{55} - x_{45}x_{52}) + x_{25}(x_{42}x_{54} - d_{44}x_{52}) \\ & - a_{12}x_{21}(d_{44}d_{55} - x_{45}x_{54}) - x_{24}(a_{41}d_{55} - x_{45}x_{51}) + x_{25}(a_{41}x_{54} - d_{44}x_{51}) + x_{14}x_{21}(x_{42}d_{55} - \\ & x_{15}x_{21}(x_{42}x_{54} - d_{44}x_{52} - d_{22}(a_{41}x_{55} - d_{44}x_{51}) + x_{24}(a_{41}x_{52} - x_{42}x_{51})) \end{aligned}$$

$$\begin{aligned} \text{Det } A(1,3,4,5) = & d_{11}d_{33}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53}) \\ & - x_{13}x_{31}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(a_{41}d_{55} - x_{45}x_{51}) + x_{25}(a_{41}x_{54} - d_{44}x_{51}) + x_{14}x_{31}(x_{43}d_{55} - \\ & x_{15}x_{31}(x_{43}x_{54} - d_{44}x_{53} - d_{33}(a_{41}x_{54} - d_{44}x_{51}) + a_{34}(a_{41}x_{53} - x_{43}x_{51})) \end{aligned}$$

$$\begin{aligned} \text{Det } A(2,3,4,5) = & d_{22}d_{33}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53}) \\ & - a_{23}x_{32}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{42}d_{55} - x_{45}x_{52}) + x_{25}(x_{42}x_{54} - d_{44}x_{52}) + x_{24}x_{32}(x_{43}d_{55} - \\ & x_{25}x_{32}(x_{43}x_{54} - d_{44}x_{53} - d_{33}(x_{42}x_{54} - d_{44}x_{52}) + a_{34}(x_{42}x_{53} - x_{43}x_{52})) \end{aligned}$$

$$\begin{aligned} \text{Det } A = & d_{11}[d_{22}(d_{33}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{43}d_{55} - x_{45}x_{53}) + x_{35}(x_{43}x_{54} - d_{44}x_{53})) \\ & - a_{23}(x_{32}(d_{44}d_{55} - x_{45}x_{54}) - a_{34}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{54} - d_{44}x_{52})) + \\ & x_{24}(x_{32}(x_{43}d_{55} - x_{45}x_{53}) - d_{33}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}x_{53} - x_{43}x_{52})) \\ & - x_{25}(x_{32}(x_{43}x_{54} - d_{44}x_{53}) - d_{33}(x_{42}x_{54} - d_{44}x_{52}) + a_{34}(x_{42}x_{53} - x_{43}x_{52}))] - \end{aligned}$$

$$a_{12}[x_{21}(d_{33}(d_{44}d_{55}-x_{45}x_{54})-a_{34}(x_{43}d_{55}-x_{45}x_{53})+x_{35}(x_{43}x_{54}-d_{44}x_{53})) - \\ a_{23}(x_{31}(d_{44}d_{55}-x_{45}x_{54})-a_{34}(a_{41}d_{55}-x_{45}x_{51})+x_{35}(a_{41}x_{54}-d_{44}x_{51})) + \\ x_{24}(x_{31}(x_{43}d_{55}-x_{45}x_{53})-d_{33}(a_{41}d_{55}-x_{45}x_{51})+x_{35}(a_{41}x_{53}-x_{43}x_{51})) - \\ x_{25}(x_{31}(x_{43}x_{54}-d_{44}x_{53})-d_{33}(a_{41}x_{54}-d_{44}x_{51})+a_{34}(a_{41}x_{53}-x_{43}x_{51}))]+$$

$$x_{13}[x_{21}(x_{32}(d_{44}d_{55}-x_{45}x_{54})-a_{34}(x_{42}d_{55}-x_{45}x_{52})+x_{35}(x_{42}x_{54}-d_{44}x_{52})) - \\ d_{22}(x_{31}(d_{44}d_{55}-x_{45}x_{54})-a_{34}(x_{41}d_{55}-x_{45}x_{51})+x_{35}(a_{41}x_{54}-d_{44}x_{51})) + \\ x_{24}(x_{31}(x_{42}d_{55}-x_{45}x_{52})-x_{32}(a_{41}d_{55}-x_{45}x_{51})+x_{35}(a_{41}x_{52}-x_{42}x_{51})) - \\ x_{25}(x_{31}(x_{42}x_{54}-d_{44}x_{52})-x_{32}(a_{41}x_{54}-d_{44}x_{51})+a_{34}(a_{41}x_{52}-x_{42}x_{51}))]-$$

$$x_{14}[x_{21}(x_{32}(x_{43}d_{55}-x_{45}x_{53})-d_{33}(x_{42}d_{55}-x_{45}x_{52})+x_{35}(x_{42}x_{53}-x_{43}x_{52})) - \\ d_{22}(x_{31}(x_{43}d_{55}-x_{45}x_{53})-d_{33}(a_{41}d_{55}-x_{45}x_{51})+x_{35}(a_{41}x_{53}-x_{43}x_{51})) + \\ a_{23}(x_{31}(x_{42}d_{55}-x_{45}x_{52})-x_{32}(a_{41}d_{55}-x_{45}x_{51})+x_{35}(a_{41}x_{52}-x_{42}x_{51})) - \\ x_{25}(x_{31}(x_{42}x_{53}-x_{43}x_{52})-x_{32}(a_{41}x_{53}-x_{43}x_{51})+d_{33}(a_{41}x_{52}-x_{42}x_{51}))]+$$

$$x_{15}[x_{21}(x_{32}(x_{43}x_{54}-d_{44}x_{53})-d_{33}(x_{42}x_{54}-d_{44}x_{52})+a_{34}(x_{42}x_{53}-x_{43}x_{52})) - \\ d_{22}(x_{31}(x_{43}d_{54}-d_{44}x_{53})-d_{33}(a_{41}x_{54}-d_{44}x_{51})+a_{34}(a_{41}x_{53}-x_{43}x_{51})) + \\ a_{23}(x_{31}(x_{42}x_{54}-d_{44}x_{52})-x_{32}(a_{41}x_{54}-d_{44}x_{51})+a_{34}(a_{41}x_{52}-x_{42}x_{51})) - \\ x_{24}(x_{31}(x_{42}x_{53}-x_{43}x_{52})-x_{32}(a_{41}x_{53}-x_{43}x_{51})+d_{33}(a_{41}x_{52}-x_{42}x_{51}))]$$

By the definition of completion, the determinants should be greater or equals to zero. Therefore, we have:

$$d_{11}d_{22} - a_{12}x_{21} \geq 0 \Rightarrow d_{11}d_{22} = a_{12}x_{21} \Rightarrow x_{21} = \frac{d_{11}d_{22}}{a_{12}}$$

$$d_{11}d_{33} - x_{13}x_{31} \geq 0 \Rightarrow d_{11}d_{33} = x_{13}x_{31} \Rightarrow x_{13} = \frac{d_{11}d_{33}}{x_{31}}$$

$$d_{11}d_{44} - x_{14}a_{41} \geq 0 \Rightarrow d_{11}d_{44} = x_{14}a_{41} \Rightarrow x_{14} = \frac{d_{11}d_{44}}{a_{41}}$$

$$d_{11}d_{55} - x_{15}x_{51} \geq 0 \Rightarrow d_{11}d_{55} = x_{15}x_{51} \Rightarrow x_{15} = \frac{d_{11}d_{55}}{x_{51}}$$

$$d_{22}d_{33} - a_{23}x_{32} \geq 0 \Rightarrow d_{22}d_{33} = a_{23}x_{32} \Rightarrow x_{32} = \frac{d_{22}d_{33}}{a_{23}}$$

$$d_{22}d_{44} - x_{24}x_{42} \geq 0 \Rightarrow d_{22}d_{44} = x_{24}x_{42} \Rightarrow x_{24} = \frac{d_{22}d_{44}}{x_{42}}$$

$$d_{22}d_{55} - x_{25}x_{52} \geq 0 \Rightarrow d_{22}d_{55} = x_{25}x_{52} \Rightarrow x_{25} = \frac{d_{22}d_{55}}{x_{52}}$$

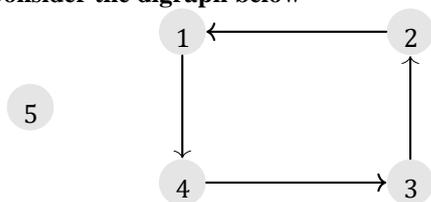
$$d_{33}d_{44} - a_{34}x_{43} \geq 0 \Rightarrow d_{33}d_{44} = a_{34}x_{43} \Rightarrow x_{43} = \frac{d_{33}d_{44}}{a_{34}}$$

$$d_{33}d_{55} - x_{35}x_{53} \geq 0 \Rightarrow d_{33}d_{55} = x_{35}x_{53} \Rightarrow x_{35} = \frac{d_{33}d_{55}}{x_{53}}$$

$$d_{44}d_{55} - x_{45}x_{54} \geq 0 \Rightarrow d_{44}d_{55} = x_{45}x_{54} \Rightarrow x_{45} = \frac{d_{44}d_{55}}{x_{54}}$$

Therefore, substituting gives the partial matrix A a positive  $P_{0,1}$  completion.

2. Consider the digraph below



From the diagram above:

Let  $A = \begin{bmatrix} d_{11} & x_{12} & x_{13} & a_{14} & x_{15} \\ a_{21} & d_{22} & x_{23} & x_{24} & x_{25} \\ x_{31} & a_{32} & d_{33} & x_{34} & x_{35} \\ x_{41} & x_{42} & a_{43} & d_{44} & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{bmatrix}$  be the partial positive  $P_{0,1}$

matrix specifying the digraph.

We have the determinants of the principal submatrices as:

$\text{Det } A(1,2) = d_{11}d_{22} - x_{12}a_{21}$  Det

$A(1,3) = d_{11}d_{33} - x_{13}x_{31}$  Det

$A(1,4) = d_{11}d_{44} - a_{14}x_{41}$  Det

$A(1,5) = d_{11}d_{55} - x_{15}x_{51}$  Det

$A(2,3) = d_{22}d_{33} - x_{23}a_{32}$

$\text{Det } A(2,4) = d_{22}d_{44} - x_{24}x_{42}$

$\text{Det } A(2,5) = d_{22}d_{55} - x_{25}x_{52}$  Det

$A(3,4) = d_{33}d_{44} - x_{34}a_{43}$

$\text{Det } A(3,5) = d_{33}d_{55} - x_{35}x_{53}$

$\text{Det } A(4,5) = d_{44}d_{55} - x_{45}x_{54}$

$\text{Det } A(1,2,3) = d_{11}(d_{22}d_{33} - x_{23}a_{32}) - x_{12}(a_{21}d_{33} - x_{23}x_{31}) + x_{13}(a_{21}a_{32} - d_{22}x_{31})$

$\text{Det } A(1,2,4) = d_{11}(d_{22}d_{44} - x_{24}x_{42}) - x_{12}(a_{21}d_{44} - x_{24}x_{41}) + a_{14}(a_{21}x_{42} - d_{22}x_{41})$

$\text{Det } A(1,2,5) = d_{11}(d_{22}d_{55} - x_{25}x_{52}) - x_{12}(a_{21}d_{55} - x_{25}x_{51}) + x_{15}(a_{21}x_{52} - d_{22}x_{51})$

$\text{Det } A(1,3,4) = d_{11}(d_{33}d_{44} - x_{34}a_{43}) - x_{13}(x_{31}d_{44} - x_{34}x_{41}) + a_{14}(x_{31}a_{43} - d_{33}x_{41})$

$\text{Det } A(1,3,5) = d_{11}(d_{33}d_{55} - x_{35}x_{53}) - x_{13}(x_{31}d_{55} - x_{35}x_{51}) + x_{15}(x_{31}x_{53} - d_{33}x_{51})$

$\text{Det } A(1,4,5) = d_{11}(d_{44}d_{55} - x_{45}x_{54}) - a_{14}(x_{41}d_{55} - x_{45}x_{51}) + x_{15}(x_{41}x_{54} - d_{44}x_{51})$

$\text{Det } A(2,3,4) = d_{22}(d_{33}d_{44} - a_{34}a_{43}) - x_{23}(a_{32}d_{44} - x_{34}x_{42}) + x_{24}(a_{32}a_{43} - d_{33}x_{42})$

$\text{Det } A(2,3,5) = d_{22}(d_{33}d_{55} - x_{35}x_{53}) - x_{23}(a_{32}d_{55} - x_{35}x_{52}) + x_{25}(a_{32}x_{53} - d_{33}x_{52})$

$\text{Det } A(2,4,5) = d_{22}(d_{44}d_{55} - x_{45}x_{54}) - x_{24}(x_{42}d_{55} - x_{45}x_{52}) + x_{25}(x_{42}x_{54} - d_{44}x_{52})$

$\text{Det } A(3,4,5) = d_{33}(d_{44}d_{55} - x_{45}x_{54}) - x_{34}(a_{43}d_{55} - x_{45}x_{53}) + x_{35}(a_{43}x_{54} - d_{44}x_{53})$

$$\text{Det A}(1,2,3,4) = d_{11}d_{22}(d_{33}d_{44}-x_{34}a_{43}) - x_{23}(a_{32}d_{44}-x_{34}x_{42}) + x_{24}(a_{32}a_{43}-d_{33}x_{12}a_{21}(d_{33}d_{44}-x_{34}a_{43}) - x_{23}(x_{31}d_{44}-x_{34}x_{41}) + x_{24}(x_{31}a_{43}-d_{33}x_{41}) + x_{13}a_{21}(a_{32}d_{44}-a_{14}a_{21}(a_{32}a_{43}-d_{33}x_{42}-d_{22}(x_{31}a_{43}-d_{33}x_{41}) + x_{23}(x_{31}x_{42}-a_{32}x_{41}))$$

$$\text{Det A}(1,2,3,5) = d_{11}d_{22}(d_{33}d_{55}-x_{35}x_{53}) - x_{23}(a_{32}d_{55}-x_{35}x_{52}) + x_{25}(a_{32}x_{53}-d_{33}x_{12}a_{21}(d_{33}d_{55}-x_{35}x_{53}) - x_{23}(x_{31}d_{55}-x_{35}x_{51}) + x_{25}(x_{31}x_{53}-d_{33}x_{51}) + x_{13}a_{21}(a_{32}d_{55}-x_{15}a_{21}(a_{32}x_{53}-d_{33}x_{52}-d_{22}(x_{31}x_{53}-d_{33}x_{51}) + x_{23}(x_{31}x_{52}-a_{32}x_{51}))$$

$$\text{Det A}(1,2,4,5) = d_{11}d_{22}(d_{44}d_{55}-x_{45}x_{54}) - x_{24}(x_{42}d_{55}-x_{45}x_{52}) + x_{25}(x_{42}x_{54}-d_{44}x_{12}a_{21}(d_{44}d_{55}-x_{45}x_{54}) - x_{24}(x_{41}d_{55}-x_{45}x_{51}) + x_{25}(x_{41}x_{54}-d_{44}x_{51}) + a_{14}a_{21}(x_{42}d_{55}-x_{15}a_{21}(x_{42}x_{54}-d_{44}x_{52}-d_{22}(x_{41}x_{55}x_{54}-d_{44}x_{51}) + x_{24}(x_{41}x_{52}-x_{42}x_{51}))$$

$$\text{Det A}(1,3,4,5) = d_{11}d_{33}(d_{44}d_{55}-x_{45}x_{54}) - x_{34}(a_{43}d_{55}-x_{45}x_{53}) + x_{35}(a_{43}x_{54}-d_{44}x_{13}x_{31}(d_{44}d_{55}-x_{45}x_{54}) - x_{34}(x_{41}d_{55}-x_{45}x_{51}) + x_{25}(x_{41}x_{54}-d_{44}x_{51}) + a_{14}x_{31}(a_{43}d_{55}-x_{15}x_{31}(a_{43}x_{54}-d_{44}x_{53}-d_{33}(x_{41}x_{54}-d_{44}x_{51}) + x_{34}(x_{41}x_{53}-a_{43}x_{51}))$$

$$\text{Det A}(2,3,4,5) = d_{22}d_{33}(d_{44}d_{55}-x_{45}x_{54}) - x_{34}(a_{43}d_{55}-x_{45}x_{53}) + x_{35}(a_{43}x_{54}-d_{44}x_{23}a_{32}(d_{44}d_{55}-x_{45}x_{54}) - x_{34}(x_{42}d_{55}-x_{45}x_{52}) + x_{25}(x_{42}x_{54}-d_{44}x_{52}) + x_{24}a_{32}(a_{43}d_{55}-x_{25}a_{32}(a_{43}x_{54}-d_{44}x_{53}-d_{33}(x_{42}x_{54}-d_{44}x_{52}) + x_{34}(x_{42}x_{53}-a_{43}x_{52}))$$

$$\text{Det A} = d_{11}[d_{22}(d_{33}(d_{44}d_{55}-x_{45}x_{54}) - x_{34}(a_{43}d_{55}-x_{45}x_{53}) + x_{35}(a_{43}x_{54}-d_{44}x_{53})) - x_{23}(a_{32}(d_{44}d_{55}-x_{45}x_{54}) - x_{34}(x_{42}d_{55}-x_{45}x_{52}) + x_{35}(x_{42}x_{54}-d_{44}x_{52})) + x_{24}(a_{32}(a_{43}d_{55}-x_{45}x_{53}) - d_{33}(x_{42}d_{55}-x_{45}x_{52}) + x_{35}(x_{42}x_{53}-a_{43}x_{52})) - x_{25}(a_{32}(a_{43}x_{54}-d_{44}x_{53}) - d_{33}(x_{42}x_{54}-d_{44}x_{52}) + x_{34}(x_{42}x_{53}-a_{43}x_{52}))]$$

$$+ x_{12}[a_{21}(d_{33}(d_{44}d_{55}-x_{45}x_{54}) - x_{34}(a_{43}d_{55}-x_{45}x_{53}) + x_{35}(a_{43}x_{54}-d_{44}x_{53})) - x_{23}(x_{31}(d_{44}d_{55}-x_{45}x_{54}) - x_{34}(x_{41}d_{55}-x_{45}x_{51}) + x_{35}(x_{41}x_{54}-d_{44}x_{51})) + x_{24}(x_{31}(a_{43}d_{55}-x_{45}x_{53}) - d_{33}(x_{41}d_{55}-x_{45}x_{51}) + x_{35}(x_{41}x_{53}-a_{43}x_{51})) - x_{25}(x_{31}(a_{43}x_{54}-d_{44}x_{53}) - d_{33}(x_{41}x_{54}-d_{44}x_{51}) + x_{34}(x_{41}x_{53}-a_{43}x_{51}))]$$

$$+ x_{13}[a_{21}(a_{32}(d_{44}d_{55}-x_{45}x_{54}) - x_{34}(x_{42}d_{55}-x_{45}x_{52}) + x_{35}(x_{42}x_{54}-d_{44}x_{52})) - d_{22}(x_{31}(d_{44}d_{55}-x_{45}x_{54}) - x_{34}(x_{41}d_{55}-x_{45}x_{51}) + x_{35}(x_{41}x_{54}-d_{44}x_{51})) + x_{24}(x_{31}(x_{42}d_{55}-x_{45}x_{52}) - a_{32}(x_{41}d_{55}-x_{45}x_{51}) + x_{35}(x_{41}x_{52}-x_{42}x_{51})) - x_{25}(x_{31}(x_{42}x_{54}-d_{44}x_{52}) - a_{32}(x_{41}x_{54}-d_{44}x_{51}) + x_{34}(x_{41}x_{52}-x_{42}x_{51}))]$$

$$+ a_{14}[a_{21}(a_{32}(a_{43}d_{55}-x_{45}x_{53}) - d_{33}(x_{42}d_{55}-x_{45}x_{52}) + x_{35}(x_{42}x_{53}-a_{43}x_{52})) - d_{22}(x_{31}(a_{43}d_{55}-x_{45}x_{53}) - d_{33}(x_{41}d_{55}-x_{45}x_{51}) + x_{35}(x_{41}x_{53}-a_{43}x_{51})) + x_{23}(x_{31}(x_{42}d_{55}-x_{45}x_{52}) - a_{32}(x_{41}d_{55}-x_{45}x_{51}) + x_{35}(x_{41}x_{52}-x_{42}x_{51})) - x_{25}(x_{31}(x_{42}x_{53}-a_{43}x_{52}) - a_{32}(x_{41}x_{53}-a_{43}x_{51}) + d_{33}(x_{41}x_{52}-x_{42}x_{51}))]$$

$$+ x_{15}[a_{21}(a_{32}(a_{43}x_{54}-d_{44}x_{53}) - d_{33}(x_{42}x_{54}-d_{44}x_{52}) + x_{34}(x_{42}x_{53}-a_{43}x_{52})) - d_{22}(x_{31}(a_{43}d_{54}-d_{44}x_{53}) - d_{33}(x_{41}x_{54}-d_{44}x_{51}) + x_{34}(x_{41}x_{53}-a_{43}x_{51})) + x_{23}(x_{31}(x_{42}x_{54}-d_{44}x_{52}) - a_{32}(x_{41}x_{54}-d_{44}x_{51}) + x_{34}(x_{41}x_{52}-x_{42}x_{51})) - x_{24}(x_{31}(x_{42}x_{53}-a_{43}x_{52}) - a_{32}(x_{41}x_{53}-a_{43}x_{51}) + d_{33}(x_{41}x_{52}-x_{42}x_{51}))]$$

By the definition of completion, the determinants should be greater or equals to zero. Therefore, we have:

$$d_{11}d_{22} - x_{12}a_{21} \geq 0 \Rightarrow d_{11}d_{22} = x_{12}a_{21} \Rightarrow x_{12} = \frac{d_{11}d_{22}}{a_{21}}$$

$$d_{11}d_{33} - x_{13}x_{31} \geq 0 \Rightarrow d_{11}d_{33} = x_{13}x_{31} \Rightarrow x_{13} = \frac{d_{11}d_{33}}{x_{31}}$$

$$d_{11}d_{44} - a_{14}x_{41} \geq 0 \Rightarrow d_{11}d_{44} = a_{14}x_{41} \Rightarrow x_{41} = \frac{d_{11}d_{44}}{a_{14}}$$

$$d_{11}d_{55} - x_{15}x_{51} \geq 0 \Rightarrow d_{11}d_{55} = x_{15}x_{51} \Rightarrow x_{15} = \frac{d_{11}d_{55}}{x_{51}}$$

$$d_{22}d_{33} - x_{23}a_{32} \geq 0 \Rightarrow d_{22}d_{33} = x_{23}a_{32} \Rightarrow x_{23} = \frac{d_{22}d_{33}}{a_{32}}$$

$$d_{22}d_{44} - x_{24}x_{42} \geq 0 \Rightarrow d_{22}d_{44} = x_{24}x_{42} \Rightarrow x_{24} = \frac{d_{22}d_{44}}{x_{42}}$$

$$d_{22}d_{55} - x_{25}x_{52} \geq 0 \Rightarrow d_{22}d_{55} = x_{25}x_{52} \Rightarrow x_{25} = \frac{d_{22}d_{55}}{x_{52}}$$

$$d_{33}d_{44} - x_{34}a_{43} \geq 0 \Rightarrow d_{33}d_{44} = x_{34}a_{43} \Rightarrow x_{34} = \frac{d_{33}d_{44}}{a_{43}}$$

$$d_{33}d_{55} - x_{35}x_{53} \geq 0 \Rightarrow d_{33}d_{55} = x_{35}x_{53} \Rightarrow x_{35} = \frac{d_{33}d_{55}}{x_{53}}$$

$$d_{44}d_{55} - x_{45}x_{54} \geq 0 \Rightarrow d_{44}d_{55} = x_{45}x_{54} \Rightarrow x_{45} = \frac{d_{44}d_{55}}{x_{54}}$$

Hence, by substitution, the partial matrix A has a positive  $P_{0,1}$  completion.

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