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Spaces Defined by Sequences of Fuzzy Sets

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Abstract: In this paper we present some definitions of concepts on fuzzy spaces which generalize the first axiom of countability. These spaces defined by sequences of fuzzy sets, for instance, sequential fuzzy space and (countably) bi- sequential fuzzy space. Following that, we consider the relationship among these spaces. The extremal topology on an arbitrary set X was defined by apazyan as a maximal non-discrete topology. In subsection, we introduce the formula of an extremal sequential fuzzy space, which is also a maximal non-discrete sequential fuzzy space. This formula depends on some fuzzy ultrafilters $\Im$. We consider some properties for this kind of topologies when $\Im$ is fixed. The subspaces and the base of this topology are also considered

Keywords: Fuzzy space, Sequential space, Ultrafilter, Extremal topology.

1. Introduction

In [1], [2], Michael and Gruenhage introduced various generalizations of first countable spaces which defined by sequences (e.g, sequential space, bi-sequential space and countably bi-sequential space) and considered the relationships among these spaces. In this paper we will discuss these properties in terms of fuzzy spaces and called these spaces, sequential fuzzy space and (countably) bi-sequential fuzzy space, where the fundamental concept of fuzzy spaces was introduced by Zadeh, see [3], which formed the backbone of fuzzy mathematics. After that some studies presented by Chang [4], again Zadeh [5], Wong [6]-[7], and Warren [8] to investigate the basic concepts and general properties of fuzzy topologies.

In 1991 the extremal topology was defined by Papazyan, see [9], which is a maximal non-discrete topology τ on a non-empty set X, so every topology which is strictly finer than τ is discrete.

On the other side, some studiesabout the concept of the extremal topologies are presented by, for instance, Sola, Abdeen and me, see [10] and [11], first on any set X and after that the studies have developed to consider this kind of topologies but on a specific kind of spaces, which are fuzzy spaces.

First of all, we will introduce some preliminary concepts related to our work. After that we will consider the property of sequentially on fuzzy spaces and introduce some examples. Following that, in a subsection, we will construct a specific formula for an extremal sequential fuzzy space $\tau_{x_\circ}$, for each $x_\circ \in X$.

In topology, a topological property is said to be hereditary if whenever a topological space has that property, then so does every subspace of it. If the latter is true only for closed subspaces or open subspace, then the property is called weakly hereditary or closed (or open)-hereditary. In this paper also we will explain how the sequentially property and the extremal sequentially property on fuzzy spaces are both weakly hereditary while the extremal property is hereditary as it explained in [11].

Later we extend our work to reach to other spaces defined by sequences, (countably) bi-sequential fuzzy spaces, and deduce the relationships among all of these spaces

2. Preliminary

The material of this section mostly comes from [3]- [7] and [12]-[15].

Let X be a non-empty set and $I = [0,1]$ unit interval, we denoted all functions from X to $I$by I^X.

Definition 2.1 A Fuzzy set A in X is every element of I^X, so a fuzzy set A in X is characterized by a membership function $\mu_A: X \to I$ which associates with each point $x \in X$ and it is denoted by $A = \{(x, \mu_A(x)): x \in X\} \subseteq X \times I$. Sometimes, we write $A(x)$for $\mu_A(x)$.

For fuzzy sets A and B in Xand $x \in X$, we have

- $A \subseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x)$.
- $\mu_{A \cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}$.
- $\mu_{A \cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$.
- $\mu_{A^c} = 1 - \mu_A$ whereA^c is the complement of A.

More generally, for a family of fuzzy sets, $\{A_i : i \in I\}$, the union $C = \bigcup_i A_i$, and the intersection, $D = \bigcap_i A_i$, are defined by $\mu_C(x) = \sup\{\mu_{A_i}(x): i \in I\}$, and $\mu(D) = \inf\{\mu_{A_i}(x): i \in I\}$.

The symbol $\tilde{0}$ will be used to denote the empty fuzzy set, so $\mu_{\tilde{0}}(x) = 0$ for all $x \in X$, whereas $\tilde{1}$ will be denoted for the fuzzy set X where $\mu_{\tilde{1}}(x) = 1$ for all $x \in X$.
For $\lambda \in (0, 1]$, a fuzzy point $P_{x_\circ}^{\lambda}$ in X is a fuzzy set in X with support $x_\circ$ and value λ, i.e:

$$P_{x_\circ}^{\lambda} = \begin{cases} \lambda, & \text{if } x = x_\circ \text{ and} \\ 0, & \text{otherwise} \end{cases}$$

so it can be written as $P_{x_\circ}^{\lambda} = \{(x_\circ, \lambda)\}$. The set of all fuzzy points of X can be denoted by $F_p(X)$.

Definition 2.2 A fuzzy topology on a set X is a family τ of fuzzy sets in X which satisfies following conditions:
- $\tilde{0}, \tilde{1} \in \tau$.
- If $A, B \in \tau$ then $A \cap B \in \tau$.
- If $A_i \in \tau$, then $\bigcup_i A_i \in \tau$.

The pair (X, τ) is called a fuzzy topological space (or fts for short) where its members are called fuzzy open sets (Fopen for short). A fuzzy set is fuzzy closed (Fclosed for short) if and only if its complement is fuzzy open. As in general topology, the indiscrete fuzzy space contains only $\tilde{0}$, and $\check{1}$, while $\widetilde{P}(X)$ the discrete fuzzy space contains all fuzzy sets and it is called the fuzzy power set of X.
For an arbitrary subset $A \subseteq X$, the induced fuzzy topology for A (or the relative fuzzy topology for A) is $\tau_A = \{A \cap U: U \in \tau: \}$. The corresponding pair (A, τ_A) is called a subspace of (X, τ). A subspace (A, τ_A) of a fts (X, τ) is called an open (closed) subspace if and only if the basic set A is τ-open (τ -closed).
A subfamily $\mathfrak{B}$ of τ is a base for τ if and only if each member of τ can be expressed as the union of some members of $\mathfrak{B}$. A fts (X, τ) is a C_1 space if and only if every fuzzy point in X has a countable local base.

Lemma 2.3 [7] If (X, τ) is C_1, then every fuzzy point p_x^r in X has a countable local base $\mathfrak{B} = \{B_n\}$ $n = 1, 2, ...$ such that
$B_1 \supseteq B_2 \supseteq B_3 \supseteq \cdots$.

Note 2.4
We write $S(A) = \{x \in X: \mu_A(x) > 0\}$ for the set of supports of the set A, where S is a function from $\widetilde{P}(X)$ to the power set $P(X)$.

Definition 2.5 [14] The usual fuzzy metric d is a function from $F_P(\mathbb{R}) \times F_P(\mathbb{R})$ to $F_P(\mathbb{R}^+)$ where $d(P_x^{\alpha}, P_y^{\lambda}) = |x - y|_{\alpha \wedge \lambda}$. The pair $(\mathbb{R}, d)$ is called the fuzzy metric space (or Fm-space for short). A fuzzy open set in $\mathbb{R}$ is $A = (a, b)_\alpha$ with $S(A) = (a, b)$ and value α. It is clear to verify that (R, d) is C_1 space where P_x^r has a countable local base $\{(x - \epsilon, x + \epsilon)\}$ for some $\epsilon > 0$ and $n \in \mathbb{Q} \cap I$.
A fuzzy Filter on a fuzzy set X is a nonempty collection $\mathfrak{F} \subset I^X$ of nonempty fuzzy sets on X which satisfies that $F_1 \cap F_2 \in \mathfrak{F}$ for all $F_1, F_2 \in \mathfrak{F}$ and if $F_1 \subseteq F_2$ and $F_1 \in \mathfrak{F}$ then $F_2 \in \mathfrak{F}$. The maximal, with respect to set inclusion, fuzzy filter on X is called a fuzzy ultrafilter on X. We say that $\mathfrak{F}$ is fixed if the intersection of all members of $\mathfrak{F}$ is nonempty, otherwise $\mathfrak{F}$ is free. A collection $\mathfrak{B} \subseteq \mathfrak{F}$ is a base for $\mathfrak{F}$ if and only if for each $F \in \mathfrak{F}$ there is some $B \in \mathfrak{B}$ such that $B \subseteq F$.

3. Sequential Fuzzy Space

Definition 3.1 A sequence of fuzzy points $\{P_{x_n}^{r_n}\}, n = 1, 2, 3, ...$ converges to a fuzzy point $P_{x_\circ}^{r_0}$, $(P_{x_n}^{r_n} \to P_{x_\circ}^{r_0})$, if and only if for every open fuzzy set U satisfies $r_0 \le \mu_U(x_\circ)$ there exists an integer m such that $r_k \le \mu_U(x_k)$, for all $k \ge m$. In this circumstances we say that $\{P_{x_n}^{r_n}\}$ is eventually in U.
Notice that, given any fuzzy point P of X and some $k \in \mathbb{N}$, then every sequence $\{P_{x_n}^{r_n}\}$ of fuzzy points such that $r_m \le \mu_P(x_m)$, for $m \ge k$, converges to P.

Definition 3.2 Let X be a fuzzy space, and $A \subseteq X$.
- A fuzzy set A is called sequentially fuzzy open (SFopen) if whenever $\{P_{x_n}^{r_n}\}$ a sequence of fuzzy points of X that converges to a fuzzy point $P_x^r \in A$ this sequence is eventually in A.
- A fuzzy set A is called sequentially fuzzy closed (SFclosed) if whenever there is a sequence of fuzzy points $\{P_{x_n}^{r_n}\}$ in A that converges to a fuzzy point P_x^r then $P_x^r \in A$,i.e. A contains the limits of all its sequences.

Note 3.3 By the definition of the convergence in a fuzzy space, every Fopen set is SFopen and every Fclosed set is SFclosed, whereas the converse is not true in general.

Corollary 3.4 A is SFclosed if and only if A^c is SFopen.

Proof. Let A be a SFclosed set and suppose that A^c is not SFopen, so A^c is not Fopen, this leads to Ais not Fclosed. Let $\wp = \{B_i : A \subseteq B_i\}, i = 1, 2, \ldots$ be the sequence of all Fclosed sets contains A, so $\overline{A} \in \wp \neq \emptyset$. Then there exists $x_\circ \in X$ such that $\bigcap_i B_i(x_\circ) > A(x_\circ)$, see [8]. We write $r_\circ = \bigcap_i B_i(x_\circ)$ and pick $P_{x_\circ}^{r_n}$ such that $r_n = B_n^c(x_\circ) \cap A(x_\circ)$. Then the sequence $\{P_{x_\circ}^{r_n}\} \subseteq A$ converges to the point $P_{x_\circ}^{r_\circ}$ which is not in A. Therefore A is not SFclose, which is a contradiction. Then A^c is SFopen.
The proof of the other side is similar, and this completes the proof.

Lemma 3.5 LetX be a fuzzy space. Then the following are equivalent:
- SFopen set is Fopen set.
- SFclosed is Fclosed set.

Proof.It is easy to prove this by using 3.3 and 3.4.

Definition 3.6 We write $\tau_{SFopenX}$ for the set of all SFopen subsets of the topological space (X, τ).

Corollary 3.7 $(X, \tau_{SFopenX})$ is a topology on X and it contains the original topology τ , that is, $\tau \subseteq \tau_{SFopenX}$.
Proof.
- It is clear that $\tilde{1}, \tilde{0} \in \tau_{SFopenX}$.
- let$\{P_{x_n}^{r_n}\}$ be a sequence of fuzzy point of X converges to $P_{x_0}^{r_0} \in U_1 \cap U_2$ where $U_1, U_2 \in \tau_{SFopenX}$. Then this sequence is eventually in U_1 and in U_2, therefore it is also eventually in $U_1 \cap U_2$. This leads to $U_1 \cap U_2 \in \tau_{SFopenX}$.
- Let $U_i \in \tau_{SFopenX}$ for all i and let $\{P_{x_n}^{r_n}\}$ be a sequence of fuzzy points converges to $P_{x_0}^{r_0} \in \bigcup_i U_i$. This sequence is eventually in U_j for some j, where $P_{x_0}^{r_0} \in U_j$, then $\{P_{x_n}^{r_n}\}$ is eventually in $\bigcup_i U_i$. Thus $\bigcup_i U_i \in \tau_{SFopenX}$.

By 3.3, every Fopen set is SFopen set, this implies that $\tau \subseteq \tau_{SFopenX}$.

Definition 3.8 A topological space (X, τ) is a sequential fuzzy space, (SFspace for short), when any fuzzy set A is Fopen if and only if it is SFopen. By 3.5, the following are equivalent:
- SFopen subset of X is Fopen set.
- SFclosed subset of X is Fclosed set.

Corollary 3.9 If (X, τ) is a sequential fuzzy space, then $\tau = \tau_{SFopenX}$.

Proof. The proof is easy.

Now we can see that the concept of the above two fuzzy sets, (SFopen, Fopen) or (SFclosed, Fclosed), are the same in some kind of spaces, for instance, the Fm-space.

Corollary 3.10 Let X be a Fm-space, then the notation of Fopen and SFopen are equivalent.

Proof.By 3.3 one side is hold. For the other side assume that A is not Fopen set, so there is $P_x^r \in A$ such that for any Fopen neighborhood $(a, b)_\alpha$ of P_x^r is not contained in A. If we pick $P_x^{r_n} \in (a, b)_{\alpha_n}$ where $0 < \alpha_n - \mu_A(x) \to 0$ for all n. It is clear to see the sequence $\{P_x^{r_n}\}$ is converges to P_x^k where $k = \mu_A(x)$ which is not eventually in A. Therefore A is not SFopen.

Remark. Let (X, τ) be fts and let R be an equivalence relation defined on X, X/R be the usual quotient set, and let ρ be the usual projection from (X, τ) onto $(X/R, \tau_R)$. $(X/R, \tau_R)$ is called the quotient fts where $\tau_R = \{U : \rho^{-1}(U) \in \tau\}$ such that ρ is continuous.
So The quotient fuzzy topology is the largest fuzzy topology such that ρ is continuous, for more explanations see [6]. The next lemma shows that a quotient space respects the property of the sequential of the original space.

Lemma 3.11 Every quotient space of a sequential fuzzy space is always a sequential fuzzy space.

Proof.Let Y be the quotient of a sequential fuzzy space X, and let ρ be a projection from X onto Y . Suppose we have a SFopen set $U \subseteq Y$, and we have a sequence $\{P_{x_n}^{r_n}\}$ of fuzzy points in X which converges to $P_x^r \in \rho^{-1}(U)$. Since ρ is continuous, so $\{\rho(P_{x_n}^{r_n})\}$ converges to $\rho(P_x^r) \in U$. Thus the sequence $\{\rho(P_{x_n}^{r_n})\}$ is eventually in U, and so $\{P_{x_n}^{r_n}\}$ is eventually in $\rho^{-1}(U)$. Then $\rho^{-1}(U)$ is SFopen in a sequential fuzzy space X, so $\rho^{-1}(U)$ is Fopen. Therefore U is Fopen in Y , and this complete the proof.

Since $(X, \tau_{SFopenX})$ has the same convergent sequences and limits as (X, τ), this leads to the following result:

Corollary 3.12. $(X, \tau_{SFopenX})$is a sequential space.

The next lemma shows the relation between the sequential fuzzy space and a C_1 space:

Lemma 3.13 Every C_1 space is a sequential fuzzy space.

Proof.Let X be a C_1 space and let A is not Fclosed set, so there exists a fuzzy point $P_x^r \in \acute{A}$ which is not in A, where $\acute{A}$ is the derived fuzzy set of A see [8]. Since Xis C_1, so P_x^r has a countable local base $\mathfrak{B} = \{B_n\}$ which satisfies $B_1 \subseteq B_2 \subseteq B_3 \subseteq \cdots$. Since $P_x^r \in \acute{A}$ this leads to $B_n \cap A \neq \tilde{0}$ for all $n \in \mathbb{N}$.
Lets choose $x_n \in X$ such that $\mu_{B_n \cap A}(x_n) > 0$ and define a sequence of fuzzy points $\{P_{x_n}^{r_n}\}$ by the following membership function: $r_n = \frac{1}{2}\mu_{B_n \cap A}(x_n)$.
It is easy to see that $\{P_{x_n}^{r_n}\}$ is in A, and it is converges to P_x^r which is not in A. This yields that A is not SFclosed. Therefore X is a sequential fuzzy space.
The converse of the above lemma is not true in general, in the following example we will give a space which is sequential whereas it is not C_1 space:

Example 3.14
Take the Fm-space (R, d), this space is C_1 where every fuzzy point P_x^r has a countable local base $\{B_n\}$ such that $S(B_n) = (x - \epsilon, x + \epsilon)$ for some $\epsilon > 0$ and has value $n \in \mathbb{Q} \cap I$ where $n \geq r$, and therefore, by 3.13, this space is a sequential fuzzy space. If we identify the set $\mathbb{Q} \cap I$ to a point, we get a quotientspace $Y = R/(\mathbb{Q} \cap I{\sim}pt)$. It is clear to verify that Y is a sequential fuzzy space, see 3.11 and 2.5, but it is not C_1.

In the following example, we will explain how the sequentiality property is weakly hereditary, i.e., we have a SFspace which has a non-sequential fuzzy subspace.

Example 3.15
Let $X = \{P_0^1\} \cup \bigcup_{i \in \mathbb{N}} X_i$
where$X_i = \{P_{1/i}^{i/(i+1)}\} \cup \{P_{x_k}^{r_k} : x_k = 1/i + 1/(i^2 + k), r_k = i/(i+1) - 1/(i^2 + k + 1), k = 0, 1, 2, \ldots\}$. It is easy to verify that $X_i \cap X_j = \emptyset$ for all $i \neq j$. Let $\{\mathbb{B}_{x_k, r_k}\} = \{P_{x_k}^{r_k}\}$ be a base for the isolated fuzzy points $P_{x_k}^{r_k}$ where x_k and r_k as above. For the point$P_{1/i}^{i/(i+1)}$ we take the base $\{\mathbb{B}_{1/i, i/(i+1)}\}$ as the family of all sets
$\{P_{1/i}^{i/(i+1)}\} \cup \{P_{x_k}^{r_k} : x_k = 1/i + 1/(j + k), r_k = i/(i+1) - 1/(j + k + 1), j = i^2, i^2 + 1, i^2 + 2, \ldots, k = 0, 1, 2, \ldots$.
Finally, for the point P_0^1 , as members of $\{\mathbb{B}_{0,1}\}$ we take all sets obtained from X by removing a finite number of X_i's and a finite number of the isolated points $(1/i + 1/t)$, of X_i's in all the remaining X_i's . one can check that $\{\mathbb{B}_{x,r}\}_{x \in X, r \in I}$ satisfies the properties of the base for a given fuzzy topological space (X, τ).
It is clear to see that $\{\mathbb{B}_{x,r}\}$ is a countable local base at P_x^r unless $\mathbb{B}_{0,1}$ which is uncountable. Thus the space (X, τ).
is not C_1 space.
supposeA is not Fclosed, this means that there is a point belongs to $\acute{A}$, the derived set of A see [8], but it dose not belong to A. If this point is $P_x^r$where $x \neq 0$, so we have a countable local base $\mathbb{B}_{x,r}$all its members meet A. We pick $P_{x_n}^{r_n} \in B_n \cap A$ for all $B_n \in \mathbb{B}_{x,r}$. Then we get a sequence $P_{x_n}^{r_n}$ in A which converges to P_x^r which is not in A, Therefore A is not SFclosed. Similarly, we can consider the case when $P_0^1 \in \acute{A} \backslash A$, and pick a subsequence $\{P_{x_n}^{r_n}\}$ from a sequence $\{P_{1/i}^{i/(i+1)}\}$ for some i. So A contains all terms of $\{P_{x_n}^{r_n}\}$ and this sequence converges to P_0^1, Therefore the space is SFspace.

Now let $Y = X\setminus\{P_{1/i}^{i/(i+1)}: i \in \mathbb{N}\}$. It is easy to verify that Y is not Fclosed in X. Define a subspace (Y, τ_Y). Thus the set $A = Y\setminus P_0^1$ together with any convergent sequence contains its limits. Since A is not Fclosed, Therefore (Y, τ_Y) is not sequential fuzzy subspace of X.

Lemma 3.16 AFclosed subspace (or Fopen subspace) of a SFspace is again SFspace.

Proof. The proof is trivial, where in these cases we have $\tau_Y \subseteq \tau_X$ such that Y is a subspace of X. For more details see [16].

3.1 Extremal Sequential Fuzzy Space

An extremal topology on a nonempty set X is a maximal non-discrete topology τ. In this section we introduce the formula of another kind of extremal topologies, which is called an extremal sequential fuzzy space and consider some properties on it.

Definition 3.17 Let X be a non-empty fuzzy set. An extremal sequential fuzzy space (X, τ) is a maximal non-discrete fuzzy space which is sequential.

Note 3.18 Let $r \in I$, throughout this paper we write

- $\widetilde{P}(X)$for the discrete fuzzy space contains all fuzzy sets of X.
- $\widetilde{P}(X|\ {}^{r}_{x_\circ})$for the collection of all fuzzy sets $A \in \widetilde{P}(X)$ where $\mu_A(x_\circ) = r$, so $\mu_{X|\ {}^{r}_{x_\circ}}(x_\circ) = r$ and otherwise $\mu_{X|\ {}^{r}_{x_\circ}}(x) = 1$.
- $\mathfrak{F} \subseteq \widetilde{P}(X|\ {}^{r}_{x_\circ})$for a fuzzy filter on X. So for any $F \in \mathfrak{F}$, $\mu_F(x_\circ) = r$.

Proposition 3.19 For any fuzzy ultrafilter $\mathfrak{F}$on X, the following are equivalent.

- $\mathfrak{F}$is fixed.
- $\mathfrak{F} = \{A \in \widetilde{P}(X): P_{x_\circ}^{r_\circ} \in A\}$, for some $x_\circ \in X$and $r_\circ \in I$.
- There exists $x_\circ \in X$ such that $P_{x_\circ}^{r_\circ} \in \mathfrak{F}$, where $r_\circ = \min\{\mu_{F_i}(x_\circ): F_i \in \mathfrak{F},\ P_{x_\circ}^{r} \in F_i\}$.

Proof.Let $\mathfrak{F}$ be a fixed ultrafilter, so $\bigcap_{F\in\mathfrak{F}} F = P_{x_\circ}^{r_\circ}$ for some $x_\circ \in X$and $r_\circ \in I$. Then $P_{x_\circ}^{r_\circ} \in F$ for all $F \in \mathfrak{F}$, this leads to $\mathfrak{F} \subseteq \{A \in \widetilde{P}(X): P_{x_\circ}^{r_\circ} \in A\}$, for some $x_\circ \in X$ and $r_\circ \in I$. Since $\mathfrak{F}$ is a fuzzy ultrafilter, then $\mathfrak{F} = \{A \in \widetilde{P}(X): P_{x_\circ}^{r_\circ} \in A\}$. If the one point set P_x^r is not in $\mathfrak{F}$ for all $x \in X$ and all $r \in I$ this contradicts $\mathfrak{F}$ is an ultrafilter, so $P_{x_\circ}^{r} \in \mathfrak{F}$ for some $x_\circ \in X$and $r \in I$. By the properties of the filters we need to choose that $r_\circ = \min\{\mu_{F_i}(x_\circ): F \in \mathfrak{F} \in, P_{x_\circ}^{r} \in F_i\}$, for some $r\in I$, to guarantee that $P_{x_\circ}^{r_\circ} \in \mathfrak{F}$. Now if the one point set $P_{x_\circ}^{r_\circ} \in \mathfrak{F}$ for some $x_\circ \in X$ and $r_\circ \in I$, so this set meets all members of the ultrafilter $\mathfrak{F}$, and this leads to $\mathfrak{F}$ is fixed.

Theorem 3.20. [11]Let X be a fuzzy space and $\mathfrak{F}$ be an ultrafilter on $X|\ {}^{0}_{x_\circ}$; i.e., $\mathfrak{F} \subseteq \widetilde{P}(X|\ {}^{0}_{x_\circ})$. Then any extremal fuzzy topology on X has the following form:$\tau_{x_\circ} = \widetilde{P}(X|\ {}^{0}_{x_\circ}) \cup \{P_{x_\circ}^{r} \cup F: F \in \mathfrak{F}\}$ for some $x_\circ \in X$and$r \in I$.

Lemma 3.21. Let $\tau_{x_\circ}$ as in 3.20, where $\mathfrak{F}$ is a fixed ultrafilter; i.e., $\bigcap_{F\in\mathfrak{F}} F = P_y^{r0}$. Then, for any $x_\circ \neq x$, $\mathfrak{B} = \{P_x^i, A : S(A) = \{x_\circ, y, z\}\}$ is a base for $\tau_{x_\circ}$, for any $z \in X - \{x_\circ, y\}$ where $i \in (0, 1]$.

Proof.It is easy to see that $\{P_x^i\}$ is a base for the part $\widetilde{P}(X|\ {}^{0}_{x_\circ})$ of the extremal fuzzy topology $\tau_{x_\circ}$. For the second part of $\tau_{x_\circ}$, since $\mathfrak{F}$ is fixed, so there is $y \in X - \{x_\circ\}$ such that $P_y^{r_\circ} \in \mathfrak{F}$. This means $P_y^{r_\circ} \in F$ for all $F \in \mathfrak{F}$. Thus, for any $z \neq x_\circ$, $\{B: S(B) = \{y, z\}\}$ is a base for $\mathfrak{F}$. Therefore $\{A : S(A) = \{x_\circ, y, z\}\}$ is a base for the part $\{P_{x_\circ}^{r} \cup F: F \in \mathfrak{F}\}$ of $\tau_{x_\circ}$, and this completes the proof.

Corollary 3.22. Let X be a non-empty fuzzy set and $\tau_{x_\circ} = \widetilde{P}(X|\ {}^{0}_{x_\circ}) \cup \{P_{x_\circ}^{r} \cup F: F \in \mathfrak{F}\}$ for some$x_\circ \in X$. If $\mathfrak{F}$ is a fixed fuzzy ultrafilter on $X|\ {}^{0}_{x_\circ}$, then $\tau_{x_\circ}$ is a C_1 space.

Proof.By 3.21, for a point $x \neq x_\circ$, $\mathfrak{B}_x = \{P_x^j\}$ is a countable local base for the fuzzy point P_x^i where $j \in \mathbb{Q} \cap I$ and $i < j$. But for the point $P_{x_\circ}^i$, we have a countable local base $\mathfrak{B}_{x_\circ} = \{A \in \widetilde{P}(X): S(A) = \{x_\circ, y\}\}$ where $\bigcap_{F\in\mathfrak{F}} F = P_y^r$, $i < \mu_A(x_\circ)$ and $r, \mu_A(x_\circ) \in \mathbb{Q} \cap I$.

Theorem 3.23. Let X be a non-empty fuzzy set. Then the extremal sequential fuzzy space has the form $\tau_{x_\circ} = \widetilde{P}(X|\ {}^{0}_{x_\circ}) \cup \{P^{r}_{x_\circ} \cup F : F \in \mathfrak{F}\}$ for some $x_\circ \in X$, where $\mathfrak{F}$ is a fixed fuzzy ultrafilter on $X|\ {}^{0}_{x_\circ}$ .i.e., $\tau_{x_\circ} = \tau_{x_\circ SFopen\ X}$.

Proof. By 3.22, $\tau_{x_\circ}$ is a C_1 space, so by 3.13 $\tau_{x_\circ}$ is a sequential fuzzy space. By 3.7, we have $\tau_{x_\circ} \subseteq \tau_{x_\circ SFopenX}$. Since $\tau_{x_\circ}$is an extremal fuzzy topology, therefore$\tau_{x_\circ} = \tau_{x_\circ SFopenX}$ is the extremal sequential fuzzy space on X.

Corollary 3.24. The nonempty Fclosed subspace (or Fopen subspace) of an extremal sequential fuzzyspace is an extremal sequential fuzzy space.

Proof.For a nonempty fuzzy subset A of Xwhere $x_\circ \in A$, we have $\tau_{A_{x_0}} = \{A \cap u : u \in \tau_{x_\circ}\} = \widetilde{P}(A|\ {}^{0}_{x_\circ}) \cup \{A \cap (P^{r}_{x_\circ} \cup F) : F \in \mathfrak{F}\} = \widetilde{P}(A|\ {}^{0}_{x_\circ}) \cup \{P^{k}_{x_\circ} \cup E : E \in \wp\}$ a subspace of an extremal sequential fuzzy space $(X, \tau_{x_\circ})$ for some $x_\circ \in X$, where $\wp \subseteq \widetilde{P}(A|\ {}^{0}_{x_\circ})$ be an ultrafilter and $k \leq r$.By [11], $\tau_{A_{x_0}}$ is an extremal topology on A. Since A is Fclosed (or Fopen), then 3.16 leads to that subspace is sequential.

4. bi- Sequential and Countably bi-sequential Fuzzy Spaces

The point $P^{r_\circ}_{x_\circ}$ is said to be an accumulation point of A if and only if for every neighborhood (or, equivalently, open neighborhood) U of $P^{r_\circ}_{x_\circ}$ one has $U \cap A_p \neq \tilde{0}$, where $A_p(x_\circ) = 0$ and otherwise it is identified with A. If $P^{r_\circ}_{x_\circ}$ is an accumulation point of A, then $P^{r_\circ}_{x_\circ}$ is an accumulation point for every other fuzzy set B with $S(A) \subseteq S(B)$.

The sequence $\{A_n\}$ of fuzzy sets is accumulating at $P^{r_\circ}_{x_\circ}$ if and only if $P^{r_\circ}_{x_\circ} \in \overline{A_n}$(the closure set of A) for all n.

Definition 4.1.A sequence of fuzzy sets $\{A_n\}$ converges to a fuzzy set A if and only if for every fuzzy open set U satisfies $\mu_A(x) \leq \mu_U(x)$ for all $x \in X$, there exists an integer m such that $\mu_{A_k}(x) \leq \mu_U(x)$, for all $k \geq m$. In this circumstances we say that $\{A_n\}$ is eventually in U.

Next we will give an example for a decreasing sequence of fuzzy sets which converges to a point:

Example 4.2. Let $X = \mathbb{R}$ with Fm-space and $A_n = \{(x, \mu_n(x)) : x \in \mathbb{R}\}$where $\mu_n(x) = e^{\frac{-nx^2}{2}}$. It is clear that $\{A_n\}$ decreasing sequence of fuzzy sets converges to P^{1}_{0}, and so P^{1}_{0} is also a cluster point.

Recall that a filter base $\mathfrak{B}$ is a non-empty collection of non-empty sets such that $F_1, F_2 \in \mathfrak{B}$ implies $F_3 \subseteq F_1 \cap F_2$ for some $F_3 \in \mathfrak{B}$. A filter base $\mathfrak{B}$ accumulates at $P^{i}_{y} \in X$ if $P^{i}_{y} \in F$ for all $F \in \mathfrak{B}$. $\mathfrak{B}$converges to P^{i}_{y} if every neighborhood of P^{i}_{y} in X contains some $F \in \mathfrak{B}$. A Filter base $\mathfrak{B}$ has cluster fuzzy point $P^{r_\circ}_{x_\circ}$ if and only if $U \cap F \neq \emptyset$ for all neighborhood U of $P^{r_\circ}_{x_\circ}$and all $F \in \mathfrak{B}$.

Definition 4.4 A fuzzy space X is called bi-sequentialfuzzy space if whenever a filter base $\mathfrak{B}$ has a cluster point $P^{r_\circ}_{x_\circ}$ in X, then there exists a decreasing sequence $\{A_n : n \in \mathbb{N}\}$ of fuzzy sets in X converges to $P^{r_\circ}_{x_\circ}$ such that $A_n \cap B \neq \tilde{0}$ for all n and all $B \in \mathfrak{B}$.

Definition 4.3 A fuzzy space X is called countably bi-sequential fuzzy space if whenever $\{A_n : n \in \mathbb{N}\}$ is a decreasing sequence of fuzzy sets accumulating at $P^{r_\circ}_{x_\circ}$ in X , there exist $P^{r_i}_{x_i} \in A_i$for each i such that $\{P^{r_n}_{x_n} : n \in \mathbb{N}\}$ converges to $P^{r_\circ}_{x_\circ}$.

Lemma 4.5 EveryC_1 space is bi-sequential.

Proof.Let $\mathfrak{B}$ be a filter base which has $P^{r_\circ}_{x_\circ}$ as a cluster point, so every neighborhood of $P^{r_\circ}_{x_\circ}$ meets all members of $\mathfrak{B}$. Since X is C_1 space, so by 2.3, $P^{r_\circ}_{x_\circ}$ has a countable local base $\{V_n\}$ such that $V_1 \supseteq V_2 \supseteq V_3 \supseteq \cdots$. Then each member of $\{V_n\}$ meets all members of $\mathfrak{B}$, therefore we get a decreasing sequence $\{V_n\}$ which converges to the point $P^{r_\circ}_{x_\circ}$, and this completes the proof.

Clearly, the next results can be deduced by the above definitions.

Lemma 4.6. Every bi-sequential fuzzy space is countably bi-sequential.

Lemma 4.7. Every countably bi-sequential fuzzy space is SFspace.

Corollary 4.8. Let X be a non-empty fuzzy set, and let $\mathfrak{F}$ be a fixed ultrafilter on $X| \ {}^{0}_{x_\circ}$. Then the extremal fuzzy space $\tau_{x_\circ} = P(X| \ {}^{0}_{x_\circ}) \cup \{P^{r}_{x_\circ} \cup F : F \in \mathfrak{F}\}$, for some $x_\circ \in X$, is bi-sequential fuzzy space.

Proof. By 3.22, $\tau_{x_\circ}$ is a C_1 space, and by the above lemmas we deduce the extremal space $\tau_{x_\circ}$ is bi-sequential fuzzy space and so is countably bi-sequential fuzzy space.

The converse of 4.6 and 4.7 is not true in general, now we will present an example for countably bi-sequential fuzzy space which is not bi-sequential but before that we will introduce some needed concepts.

Remark [17]-[18].The pair $(X, \sum)$ is called a measurable space where a σ-algebra on a set X is a collection $\sum$ of subsets of X that includes X itself, is closed under complement, and is closed under countable unions.
A function γ from $\sum$ to the extended real number line is called a measure if it satisfies the properties of non-negativity, null empty set and countable additivity.
The classical concept of measure considers that $\sum \subseteq \{0,1\}^X$ but this consideration can be extendedto a set of fuzzy subsets $\mathfrak{F}$of X, $\mathfrak{F} \subseteq [0,1]^X$, satisfying the properties of measurable space $([0,1]^X, \mathfrak{F})$.
In the classical definition of measure we use additive property. Additivity is very effective in many applications, but in many real world problems we do not require measure with respect to the additive feature, for example in fuzzy logic.
The cardinal of X, (card X), is measurable if there exists a countably additive non-trivial measure on P(X) taking only the value 0, 1 where X has measure 1 and the points have measure 0. There is a relationship between measurable cardinal and existence of some ultrafilters on X with some properties. The following lemma shows this result and the reader can find the proof in [19].

Lemma 4.9 Card X is measurable if and only if there is an ultrafilter $\mathfrak{F}$ on X such that $\cap_{F\in\mathfrak{F}} F = \tilde{0}$ but $\cap_{E\in\wp} E \in \mathfrak{F}$ for every countable $\wp \subseteq \mathfrak{F}$.
Each non compact, locally compact space X has a compactification $X^* = X \cup \{pt\}$ with one point remainder, its topology has the form $\tau_* = \tau \cup \{U \cup \{pt\} : U \in \tau\}$ such that U^c is compact in X. For more explanations see [20], and see [21] in terms of fuzzy spaces.
The following example shows that the converse of 4.6 is not true in general.

Example 4.10 Let X be an infinite discrete fuzzy space and let $X^* = X \cup \{P\}$be the one point compactification of X for fuzzy point P not in X. Then

- $X^*$is countable bi-sequential fuzzy space.
- If card X^* (= Card X) is measurable then X^* is not bi-sequential fuzzy space.

For the first part of the example, let $\{A_n\}$ be a decreasing sequence of fuzzy sets accumulating at $P^{r_\circ}_{x_\circ}$, so $P^{r_\circ}_{x_\circ} \in \overline{A_n}$ for all $n \in \mathbb{N}$. If $P^{r_\circ}_{x_\circ} \in A_n$ for all n, we can pick $P^{r_n}_{x_n} \in A_n$ for all n and this leads to the requirement. But if no, thus we assume that $P^{r_\circ}_{x_\circ} = P$ where A_n are all infinite, therefore we choose $P^{r_n}_{x_n} \in A_n$ which also leads to the requirement.
Now to show this space is not bi-sequential. By 4.9, we have an ultrafilter $\mathfrak{F}$ on X^* such that $\bigcap_{F\in\mathfrak{F}} F = \tilde{0}$. Let $\mathfrak{B}$ be a filter base for $\mathfrak{F}$, so also $\bigcap_{B\in\mathfrak{B}} B = \tilde{0}$. , this implies that $\mathfrak{B}$ converges to P, and so it is a cluster point of the filter base. By 4.9, if we choose $\wp \subset \mathfrak{F}$ to be decreasing sequence such that$\bigcap_{E\in\wp} E \in \mathfrak{F}$. Then $\bigcap_{E\in\wp} E \cap X \neq \tilde{0}$. , so $\wp$ does not converge to P.ThusX^* is not bi-sequential fuzzy space.

5. Conclusion

Using the converging of the sequences of fuzzy subsets of a spaceX we introduce the definitions of some spaces called sequential fuzzy space, bi-sequential fuzzy space and countably bi-sequential fuzzy space. The relationships among these spaces and the C_1 space is explained in the following diagram:

C_1space⟹ bi-sequential fuzzy space ⟹ countably bi-sequential fuzzy space ⟹ sequential fuzzy space.

The converse ofall arrows in the above diagram are not true in general, so we gave some examples to explain that.

The extremal sequential fuzzy spaces took a share as a part of this paper, and conclude that the sequentially property and the extremal sequentially property on fuzzy spaces are both weakly hereditary while the extremal property is hereditary.

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