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On The Transcendental Equation $\sqrt[3]{x^2+y^2}+\sqrt[2]{mx+ny}=10z^3$

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Abstract: The transcendental equation with five unknowns involving surds represented by the diophantine equation $\sqrt[3]{x^2+y^2}+\sqrt[2]{mx+ny}=10z^3$ is analysed for its patterns of non-zero distinct solutions.

Keywords: Transcendental equation, integral solutions, surd equation.

1. Introduction

Diophantine equations have an unlimited field of research by reason of their variety. Most of the Diophantine problems are algebraic equations [1,2]. In [3-17], the integral solutions of transcendental equations involving surds are analyzed for their respective integer solutions. This communication analyses a transcendental equation with five unknowns given by $\sqrt[3]{x^2+y^2}+\sqrt[2]{mx+ny}=10z^3$. Infinitely many non-zero integer quintuples (x,y,z,m,n) satisfying the above equation are obtained.

2. Method of analysis

The transcendental equation involving surds to be solved is

$$\sqrt[3]{x^2+y^2}+\sqrt[2]{mx+ny}=10z^3 \tag{1}$$

The introduction of the transformations

$$x=m(m^2+n^2) \quad , \quad y=n(m^2+n^2) \tag{2}$$

in (1) leads to

$$m^2+n^2=5z^3 \tag{3}$$

To start with, observe that

$$m=2\alpha^{3k}\,,\; n=\alpha^{3k}\,,\; z=\alpha^{2k} \tag{4}$$

Satisfy (3). In view of (2), one obtains

$$x=10\alpha^{9k} \quad , \quad y=5\alpha^{9k} \tag{5}$$

Thus, the quintuple (x,y,z,m,n) given by $(10\alpha^{9k},5\alpha^{9k},\alpha^{2k},2\alpha^{3k},\alpha^{3k})$ satisfies (1).

Also, taking

$$m=5^2M\,,\; n=5^2N\,,\; z=5\alpha^2 \tag{6}$$

in (3), it is written as

$$M^2+N^2=(\alpha^3)^2 \tag{7}$$

which is satisfied by

$$M=2uv\,,\; N=u^2-v^2\,, \qquad u>v>0 \tag{8}$$

$$\alpha^3=u^2+v^2 \tag{9}$$

Again, note that (9) is satisfied by

$$u=p(p^2+q^2),\; v=q(p^2+q^2),\; \alpha=p^2+q^2\,, \quad p>q>0 \tag{10}$$

From (10), (8) and (6), one obtains

$$\left.\begin{aligned} m&=50pq(p^2+q^2)^2 \\ n&=25(p^2-q^2)(p^2+q^2)^2 \end{aligned}\right\} \tag{11}$$

$$z = 5\left(p^2 + q^2\right)^2 \quad (12)$$

Substituting (11) in (2), it is seen that

$$\left.\begin{array}{l} x = 2 * 25^3\, pq\left(p^2 + q^2\right)^8 \\ y = 25^3\left(p^2 - q^2\right)\left(p^2 + q^2\right)^8 \end{array}\right\} \quad (13)$$

Thus, (11), (12) and (13) satisfy (1).

Note 1:
It is worth to note that (9) is also satisfied by

$$u = m^3 - 3mn^2\ , \quad v = 3m^2n - n^3\ , \quad \alpha = m^2 + n^2$$

Proceeding as above, a different set of solutions to (1) is obtained.

Note 2:
Observe that (7) may be written as

$$M^2 + N^2 = \left(\alpha^2\right)^3$$

which is satisfied by

$$\left.\begin{array}{l} M = p\left(p^2 + q^2\right) \quad , \quad p^3 - 3pq^2 \\ N = q\left(p^2 + q^2\right) \quad , \quad 3p^2q - q^3 \end{array}\right\} \quad (14)$$

$$\alpha^2 = p^2 + q^2 \quad (15)$$

It is well-know that (15) represents the Pythagorean equation whose solution may be written as

$$p = 2ab \quad , \quad q = a^2 - b^2 \quad , \quad \alpha = a^2 + b^2, \quad a > b > 0$$

In view of (6) and (2), two different sets of integer solutions to (1) are obtained.
In addition to the above sets of integer solutions to (1), there are other sets of integer solutions to (1) which are illustrated below:
Set 1: Assume

$$z = a^2 + b^2 \quad (16)$$

write 5 as

$$5 = (2 + i)(2 - i) \quad (17)$$

Substituting (16), (17) in (3) and applying the method of factorization, define

$$m + in = (2 + i)(a + ib)^3$$

from which we get

$$\left.\begin{array}{l} m = 2a^3 - 6ab^2 - 3a^2b + b^3 \\ n = a^3 - 3ab^2 + 6a^2b - 2b^3 \end{array}\right\} \quad (18)$$

In view of (2), it is seen that

$$\left.\begin{array}{l} x = 5\left(a^2 + b^2\right)^3\left(2a^3 - 6ab^2 - 3a^2b + b^3\right) \\ y = 5\left(a^2 + b^2\right)\ \left(a^3 + 6a^2b - 3ab^2 - 2b^3\right) \end{array}\right\} \quad (19)$$

Thus, (16), (18) and (19) represent the required integer solutions to (1).

Note 3:
One may also written 5 as

$$5 = (1 + 2i)(1 - 2i) \quad (20)$$

In this case, the corresponding integer solutions to (1) are found to be

$$x = 5\left(a^2 + b^2\right)^3\left(a^3 - 3ab^2 - 6a^2b + 2b^3\right)$$

$$y = 5\left(a^2 + b^2\right)^3\left(2a^3 - 6ab^2 + 3a^2b - b^3\right)$$

$$z = a^2 + b^2$$

$$m = a^3 - 3ab^2 - 6a^2b + 2b^3$$

$$n = 2a^3 - 6ab^2 + 3a^2b - b^3$$

Set 2: (3) is written as

$$m^2 + n^2 = 5z^3 * 1 \quad (21)$$

Consider 1 to be

$$1=\frac{(p^2-q^2+i2pq)(p^2-q^2-i2pq)}{(p^2+q^2)^2}$$

(22)

(or)

$$1=\frac{(2pq+i(p^2-q^2))(2pq-i(p^2-q^2))}{(p^2+q^2)^2} \quad (23)$$

Substituting (16), (17) and (22) in (21) and applying the method of factorization, define

$$m+in=(2+i)(a+ib)^3\frac{(p^2-q^2+i2pq)}{(p^2+q^2)}$$

On equating the real and imaginary parts, we get

$$m=\frac{1}{(p^2+q^2)}\left(f(p,q)(a^3-3ab^2)-(3a^2b-b^3)g(p,q)\right)$$

$$n=\frac{1}{(p^2+q^2)}\left((a^3-3ab^2)g(p,q)+(3a^2b-b^3)f(p,q)\right)$$

where

$$f(p,q)=2p^2-2q^2-2pq \quad , \quad g(p,q)=p^2-q^2+4pq$$

As our interest is on finding integer solutions, replacing a by $(p^2+q^2)A$ and b by $(p^2+q^2)B$, we have

$$\left.\begin{aligned} m&=(p^2+q^2)^2\left((A^3-3AB^2)f(p,q)-(3A^2B-B^3)g(p,q)\right)\\ n&=(p^2+q^2)^2\left((A^3-3AB^2)g(p,q)+(3A^2B-B^3)f(p,q)\right)\\ z&=(p^2+q^2)^2(A^2+B^2)\end{aligned}\right\} \quad (24)$$

In view of (2), one obtains

$$\left.\begin{aligned} x&=5(A^2+B^2)^3(p^2+q^2)^8\left((A^3-3AB^2)f(p,q)-(3A^2B-B^3)g(p,q)\right)\\ y&=5(A^2+B^2)^3(p^2+q^2)^8\left((A^3-3AB^2)g(p,q)+(3A^2B-B^3)f(p,q)\right)\end{aligned}\right\} \quad (25)$$

Thus, (24) and (25) represent the integer solutions to (1).

Note 4:

In addition to (17) and (22), one may also consider the values of 5 and 1 in (21) to be represented by the choices of equations (17), (23); (20), (22); (20), (23) respectively and thus, the other sets of integer solutions to (1) are obtained.

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