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Quantum Equation for Generalized Special Relativistic Linear Hamiltonian

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Abstract: Using generalized special relativistic energy –momentum relation a useful linear equation was obtained .the coefficients and matrixes resembles that of Dirac relativistic quantum equation .Anew quantum linear relativistic equation sensitive to the potential and the effects of fields was also obtained. This equation reduces to that of Dirac in the absence of fields. The perturbed Hamiltonian consist of free energy term beside linear potential term which resembles that of ordinary perturbation theories. The travelling wave solution gives anew potential dependent energy relation, which reduces to that of Dirac in the absences of field. Move over this expression for energy can be a pure imaginary for strong potential and energetic particle, which indicates efficient energy absorption by the medium as proposed by electromagnetic theory.

Keywords: Generalized special relativity, Dirac equation, perturbed Hamilton, Energy –momentum relation, potential

Introduction

The behavior of particles in our world is mainly described by Newton’s laws. However Newton’s lows cannot describe the atomic world, which can be explained by using quantum physics [1, 2, 3]. Though quantum effects are seen in sub atomic level, it is equally good for particles and system of all order. Newton’s laws are fundamental laws that describe the energy evolution of any system laws in Newtonian mechanics. Its energy relation is used to construct Schrodinger quantum equation. It describes the state of atomic and subatomic systems, like elementary particles that moves with relatively law speed compared to light speed. On the other hand Plank introduced the quantum concept and plank’s constant where he assumed that electromagnetic radiation exists only in discrete energy states, where electromagnetic waves behaves as discrete particles. One of the main notions of quantum laws is the wave particle duality [3, 4, 5].To describe high speed particles two relativistic quantum equations are proposed , one is the Klein –Gordon equation ,the other is Dirac equation .The former describes spin less particles while the latter describes spinning particles [5,6]. Desibe their remarkable successes, they suffer from being un capable of describing fields other than electromagnetic fields. Some attempts were made for Klein-Gordon equation to cure this defect [7, 8, 9]. But one needs to extend it to cure Dirac equation defects also. Such attempt was done in section two.

Perturbed Hamiltonian for linear generalized special relativity

Use Generalized Special Relativistic Energy Quantum equation to find time departures in energy space by assuming the wave function to be

$$\psi(r,t) = \sum_{n=1}^{\infty} C_n(t)\psi_n^0(r) \qquad (1)$$

And the GSR energy to be given by

$$E = \frac{g_{00} m_o c^2}{\sqrt{g_{00} - \frac{m^2 v^2 c^2}{m^2 c^4}}}$$

$$E = \frac{g_{00} m_o c^2 E}{\sqrt{g_{00}E^2 - p^2c^2}}$$

$$g_{00}E^2 = g_{00}^2 m_o^2 c^4 + p^2c^2 \qquad (2)$$

$$g_{00} = 1 + \frac{2\phi}{c^2} \qquad (3)$$

When: $\phi = 0$

$$E^2 = m_o^2 c^4 + p^2c^2 \qquad (4)$$

The Linear form of (E) can be found by rearranging equation (2)to be:

$$E^2 = g_{00} m_o^2 c^4 + g_{00}^{-1} p^2c^2 \qquad (5)$$

Consider linear

$$E = g_{00}^{1/2}\, \beta m_o c^2 + g_{00}^{-1/2} c\alpha \,.\, p \qquad (6)$$

For $\phi = 0$: $g_{00} = 1$

The linear energy E GSR reduced to that of Dirac Relativistic equation

$$E = \beta m_o c^2 + c\alpha \,.\, p \qquad (7)$$

The corresponding Hamiltonian can thus be written as

$$\widehat{H}_0 = \beta m_o c^2 + c\alpha \,.\, \hat{p}$$

$$\widehat{H}_0 = \frac{ch}{i} \alpha \cdot \nabla + \beta m_o c^2 \qquad (8)$$

For

$$\phi \neq 0 \quad \text{and} \quad g_{00} = 1 + \frac{2\phi}{c^2} \qquad (9)$$

For

$$\phi \ll c^2 \quad : \quad g_{00}^{1/2} = (1 + \frac{2\phi}{c^2})^{1/2}$$

$$g_{00}^{1/2} = 1 + \frac{1}{2}(\frac{2\phi}{c^2}) = 1 + \frac{2\phi}{c^2} = 1 + \frac{2m_0\,\phi}{2m_0\,c^2} = 1 + \frac{V_0}{m_0\,c^2}$$

$$= 1 + \frac{V}{m_0\,c^2} \qquad (10)$$

$$g_{00}^{-1/2} = (1 + \frac{2\phi}{c^2})^{-1/2} = 1 + (-\frac{1}{2})(\frac{2\phi}{c^2}) = 1 - (\frac{2\phi}{c^2}) = 1 - \frac{\phi}{c^2}$$

$$= 1 - \frac{m_0\,\phi}{m_0\,c^2} = 1 - \frac{V}{m_0\,c^2} \qquad (11)$$

The Hamiltonian of a perturbed system is given by considering the energy found from(10), (11) and (6)

$$E = g_{00}^{1/2}\, \beta m_o c^2 + g_{00}^{-1/2} c\alpha \,.\, p$$

$$E = \beta m_o c^2 (1 + \frac{V}{m_0\,c^2}) + c\,(1 - \frac{V}{m_0\,c^2})\,\alpha \cdot P$$

$$= \beta m_o c^2 + c\,\alpha \,.\, p + \beta V - \frac{Vc}{m_0\,c^2} \alpha \cdot P \qquad (12)$$

Let $P \cong m_0\, c$

$$E = E_0 + (\beta - \alpha)V \qquad (13)$$

Thus from (13) the Hamiltonian is comes:

$$H = H_0 + (\beta - \alpha)V = H_0 + H_1$$

Using transformation

$$E \to H \quad , \quad E_0 \to H_0 \quad \& \quad (\beta - \alpha)V \equiv H_1 \qquad (14)$$

Thus the perturbation term of the Hamiltonian is given by

$$\widehat{H}_1 = (\beta - \alpha)V \qquad (15)$$

Generalized special Relativistic linear Hamiltonian

For SR the liner energy from is given by:

$$E = \beta m_0 c^2 + c\alpha \,.\, p \qquad (16)$$

For GSR the energy expiration is given by:

$$E^2 = g_{00} m_o^2 c^4 + g_{00}^{-1} p^2c^2 \qquad (17)$$

In analogy with that of SR, the GSR linear energy is given by

$$E = g_{00}^{1/2}\, \beta m_o c^2 + g_{00}^{-1/2} c\alpha \,.\, p \qquad (18)$$

Required to find α and β square both sides of equation (18)

$$E^2 = (g_{00}^{1/2}\, \beta m_o c^2 + g_{00}^{-1/2} c\alpha \,.\, p)\,(g_{00}^{1/2}\, \beta m_o c^2 + g_{00}^{-1/2} c\alpha \,.\, p) \qquad (19)$$

$$= (\beta^2 g_{00} m_o^2 c^4 + c^3 \beta\alpha \,.\, p m_0 + c^3 \alpha \,.\, p m_0 \beta + c^2 g_{00}{}^{-1} \alpha^2 \,.\, p^2\,) \qquad (20)$$

$$= \beta^2 g_{00} m_o^2 c^4 + c^2 g_{00}{}^{-1} \alpha^2 \,.\, p^2 + \; m_0 c^3 (\beta\alpha \,.\, p + \alpha \,.\, p\beta)$$

$$E^2 = \beta^2 g_{00} m_o^2 c^4 + c^2 g_{00}{}^{-1} (\alpha \,.\, p)^2 + m_0 c^3 (\beta\alpha \,.\, p + \alpha \,.\, p\beta) \qquad (21)$$

Expressing both (α and p) in three dimensions, one gets

$E^2 = \beta^2 g_{00} m_o^2 c^4 + g_{00}{}^{-1} c^2 ((\alpha_x.\, p_x + \alpha_y.\, p_y + \alpha_z.\, p_z)(\alpha_x.\, p_x + \alpha_y.\, p_y + \alpha_z p_z)) + \; m_0 c^3 (\beta(\alpha_x.\, p_x + \alpha_y.\, p_y + \alpha_z.\, p_z) +$ (αx.px+αy.py+αz.pz)β

$E^2 = \beta^2 g_{00} m_o^2 c^4 + g_{00}{}^{-1} c^2 ((\alpha_x.\, p_x + \alpha_y.\, p_y + \alpha_z.\, p_z)(\alpha_x.\, p_x + \alpha_y.\, p_y \alpha_z p_z)) + \; m_0 c^3 \,((\beta\alpha_x \,.\, p_x + \alpha_{\,x}.\, p_x \beta) +$ βαy .py+α y.pyβ+βαz.pz+α z.pzβ (22)

But

$$p_x\beta = \beta\, p_x \quad , \quad p_y\beta = \beta\, p_y \quad , \quad p_z\beta = \beta\, p_z \qquad (23)$$

Because

$$p_x\beta\, \psi = (\frac{\hbar}{i}\frac{\partial}{\partial x})\; \beta\psi = \beta\, (\frac{\hbar}{i}\frac{\partial}{\partial x})\, \psi = \beta p_x \psi$$

$$p_y\beta\, \psi = (\frac{\hbar}{i}\frac{\partial}{\partial y})\; \beta\psi = \beta\, (\frac{\hbar}{i}\frac{\partial}{\partial y})\, \psi\beta = p_y \psi$$

$$p_z\beta\, \psi = (\frac{\hbar}{i}\frac{\partial}{\partial z})\; \beta\psi = \beta\, (\frac{\hbar}{i}\frac{\partial}{\partial z})\, \psi = \beta p_z \psi$$

$\therefore\;\; E^2 = \beta^2 g_{00} m_o^2 c^4 + g_{00}{}^{-1} c^2 (\alpha_x{}^2 p_x{}^2 + \alpha_y{}^2 p_y{}^2 + \alpha_z{}^2 p_z{}^2 + \alpha_x p_x \alpha_y p_y + \alpha_x p_x \alpha_z p_z + \alpha_{\,y} p_y \alpha_x p_x + \alpha_{\,y} p_y \alpha_z p_z +$ α zpzαxpx+α zpzαypy++ m0c3βαx px+α xβpx+βαy py+α yβpy+βαz pz+α zβpz (24)

$E^2 = \beta^2 g_{00} m_o^2 c^4 + g_{00}{}^{-1} c^2 (\alpha_x{}^2 p_x{}^2 + \alpha_y{}^2 p_y{}^2 + \alpha_z{}^2 p_z{}^2 + (\alpha_x\alpha_y + \alpha_y\alpha_y) p_x p_y + (\alpha_x\alpha_z + \alpha_z\alpha_x) p_x p_z +$ αyαz+αzαypypz+ m0c3βαx +α xβpx+(βαy +α yβ)py+(βαz +α zβ)pz 25

Re writing equation (17) in three dimensions yields:

$$E^2 = g_{00} m_o^2 c^4 + g_{00}{}^{-1} c^2 (p_x{}^2 + p_y{}^2 + p_z{}^2) \qquad (26)$$

Comparing with equation (25) and (26) one finds that

$\beta^2 = 1 \qquad \alpha_x{}^2 = 1 \qquad \alpha_y{}^2 = 1 \qquad \alpha_z{}^2 = 1$

$\alpha_x\alpha_y + \alpha_y\alpha_x = 0 \qquad \alpha_x\alpha_z + \alpha_z\alpha_x = 0 \qquad \alpha_y\alpha_z + \alpha_z\alpha_{\,y} = 0 \qquad (27)$

Fortunately these relations are typical to that of Dirac quantum relativistic equation

Hence

$$E = g_{00}^{1/2}\, \beta m_o c^2 + g_{00}^{-1/2} c\alpha \,.\, p \qquad (28)$$

The corresponding linear quantum mechanical Hamiltonian can be found by adopting the transformation

$$E \to \hat{H} \;= i\hbar\frac{\partial}{\partial t} \quad \text{and} \quad p = \hat{p} = \frac{\hbar}{i}\nabla \qquad (29)$$

Inserting (28) in (29) yields

$$\hat{H} = g_{00}^{-1/2} c\alpha \,.\, \hat{p} + g_{00}^{1/2}\, \beta m_o c^2 \qquad (30)$$

$$i\hbar\frac{\partial}{\partial t} = c g_{00}^{-1/2} \frac{\hbar}{i} \alpha \,.\, \nabla \; + g_{00}^{1/2}\, \beta m_o c^2 \qquad (31)$$

Acting on ψ gives

$$i\hbar\frac{\partial\psi}{\partial t} = c g_{00}^{-1/2} \frac{\hbar}{i} \alpha \,.\, \nabla\psi \; + g_{00}^{1/2}\, \beta m_o c^2\, \psi \qquad (32)$$

but $g_{00} = 1 + \frac{2\phi}{c^2} \;\to\; g_{00}^{1/2} = 1 + \frac{\phi}{c^2} \quad , \quad g_{00}^{-1/2} = 1 - \frac{\phi}{c^2} \qquad (33)$

But (29) in (28) find

$$i\hbar\frac{\partial\psi}{\partial t} = c\left(1 - \frac{\phi}{c^2}\right)\frac{\hbar}{i}\alpha \,.\, \vec{\nabla}\psi \; + \left(1 + \frac{\phi}{c^2}\right)\, \beta m_o c^2\, \psi \qquad (34)$$

This is the GSR Dirac equation. Another non linear Dirac equation can be derived by using the relations

$$E = mc^2 \qquad v = m\,\phi \qquad (35)$$

In equation (28) together with equation (33) to get

$$E\psi = c\,(1-\frac{v}{E})\,\alpha\,.\,p\psi\ +(1+\frac{v}{E})\ \beta m_o c^2\,\psi \qquad (36)$$

$$E^2\psi = cE\,(1-\frac{v}{E})\,\alpha\,.\,p\psi\ +E\,(1+\frac{v}{E})\ \beta m_o c^2\,\psi$$

$$\therefore E^2\psi = c(E-v)\alpha\,.\,p\psi\ +(E+v)\ \beta m_o c^2\,\psi \qquad (37)$$

$$E^2\psi = c\alpha\,.\,pE\psi - cv\alpha\,.\,p\psi + \beta m_o c^2\,E\psi + \beta m_o c^2\,v\psi$$

Thus from equation (29)

$$-\hbar^2\frac{\partial^2\psi}{\partial t^2} = c\hbar^2\alpha.\nabla\ (\frac{\partial\psi}{\partial t}) - c\frac{\hbar}{i}\ v\,\alpha.\nabla\psi + i\hbar\ \beta m_o c^2\ (\frac{\partial\psi}{\partial t}) + \ \beta m_o c^2\,v\,\psi \ (38)$$

Which is anon linear Dirac equation?

A travelling wave solution can be found by suggesting

$$\psi = e^{-i\omega t}u \qquad (39)$$

And substituting in (38) to get

$$\hbar^2\omega^2 e^{-i\omega t}\,u(\underline{r}) =$$
$$e^{-i\omega t}\,(-c\hbar^2\alpha.\,i\omega\,\nabla\,u(\underline{r}) - \ c\frac{\hbar}{i}\ v\,\alpha.\nabla\,u(\underline{r}) + \hbar\omega\beta m_o c^2\,u(\underline{r}) +$$
$$\beta moc2\ v\ u\underline{r} \qquad (40)$$

Try now the expression

$$u(\underline{r}) = Ae^{ikx} = u \quad , \quad \nabla u(\underline{r}) = ikAe^{ikx} \quad ,\nabla u(\underline{r}) = iku \qquad (41)$$

Put (41) in (40) to get

$$\hbar^2\omega^2 e^{-i\omega t}\,u\ = e^{-i\omega t}(c\hbar^2\alpha.\,\omega\,ku - \ c\hbar\,v\,\alpha.\,ku + \hbar\omega\ \beta m_o c^2\,u\ +\ \beta m_o c^2\,vu\,) \qquad (42)$$

Since:

$$E = \hbar\omega\ ,\ \ P = \hbar k\ \ ,\ \ E_0 = m_0c^2 \qquad (43)$$

It follows that equation (43) gives

$$E^2 = cE\alpha.\,P - cv\alpha.\,P + E\beta E_0 + \beta E_0 v$$

$$E^2 - E(c\alpha.\,P + \beta E_0) + v(c\alpha.\,P - \beta E_0) = 0 \qquad (44)$$

Where

$$ax^2 + bx + c = 0$$

$$a = 1 \quad b = -(c\alpha.\,P + \beta E_0) \quad c = (c\alpha.\,P - \beta E_0)v \qquad (45)$$

$$Thus\ \ b = \ E_D \qquad D \equiv \text{Dirac}$$

$$x = E = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad = \frac{E_D \pm \sqrt{{E_D}^2 + 4(\beta E_0 - c\alpha.\,P)v}}{2} \qquad (46)$$

When no potential exists:

$$E = \ E_D$$

Which is the ordinary Dirac linear relativistic energy, or $E = 0$
Which corresponding to free space

Discussion

The linear GSR energy –momentum equation (12) which was derived from GSR energy –momentum relation for a weak field, is unlike that of Einstein –Dirac. This is due to the fact that it consists of additional term representing the potential. In Einstein –Dirac relation the energy of a particle moving in gravitational or nuclear field is the same as that of a particle moving in free space with the same speed. This is in direct conflict with observations, where the spectra of free electrons are different from the spectra of atomic electrons. The spectra of atomic electrons themselves are strongly dependent on the nuclear potential. This means that our equation (12) is more realistic than that of SR, as far as it conforms with experiments. The linear form of E in GSR corresponding to that of Dirac is proposed and its form is shows in equation (18). Fortunately the mathematical analysis shows that the coefficients α and β are typical to that of Einstein-Dirac coefficients. Using this linear form in equation (18),a useful linear GSR Dirac equation has been derived to get equation (31),(34),which is reduced to the conventional Dirac one in the absence of any field ,where ($g_{00} \to 1$). Another non linear form can be derived by suggesting $V = m\phi$, $E = mc^2$ to get non linear equation in the time derivative, beside a mixed time and spatial differential term [see equation (38)].This equation is also consists of a potential term which can discriminate between vacuum and atoms having different nuclear potential and different spectra. The travelling wave solution in equations (39) and (41) shows that the energy is given by (39) and (41). It is very interesting to note that in the absence of fields the energy reduces to that of Dirac, However for very strong field, when the particle is highly energetic, such that $c\alpha.p > \beta E_0$ .The energy becomes purely imaginary. This may be explained by bearing in mind that energetic particles causes high loss in a medium. This conforms with the fact that for travelling waves the imaginary energy reflects absorption according to the relation

$$E = E_0 + iE_a$$

$$P = P_0 + iP_a = \frac{E_0}{c} + \frac{iE_a}{c}$$

$$\psi = Ae^{i(kx - \omega t)}$$

$$\psi = Ae^{-\frac{E_a}{c}x} e^{i(\frac{iE_0}{c}x - \omega t)}$$

$$I = |\psi|^2 = A^2 e^{-\frac{2E_a}{c}x} = I_0 e^{-\alpha x}$$

Thus the absorption coefficient is given by

$$\alpha = \frac{2E_a}{c}$$

Conclusion

The coefficient and matrixes of the GSR linear Dirac quantum equation resembles that of Dirac. The GSR linear relativistic equation consists of an additional term, representing potential. This equation reduces to that of ordinary Dirac in the absence of fields. The travelling wave solution gives an energy expression reduces to that of Dirac in the absence of fields.

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