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Analytical solutions of nonlinear time-fractional Hirota-Satsuma equations using the improved F-expansion method

Md. Anisur Rahman Pramanik[a], Manik Mondal[b], Md. Shahin Alam[c]
[a] *Department of Computer Science & Engineering, Dhaka International University, Dhaka, Bangladesh.*
bDoctoral student, Nagaoka University of Technology, Nagaoka, Niigata, Japan.
[c]*Department of Mechanical , Sonargoan University, Dhaka, Bangladesh.*

Abstract: In this paper, we implement the improved F-expansion method with Riccati equation for finding the analytical solutions of the nonlinear time-fractional Hirota-Satsuma equation. We have successfully established three classes of explicit solutions-hyperbolic, trigonometric and rational solutions with some free parameters. For particular values of the parameters, solitary wave solutions are started from the travelling wave solutions. Our results release that this method is straightforward, concise, very active and is a promising and powerful method for other nonlinear evolution equations arising in mathematical physics and engineering.

Keywords: Homogeneous balance; improved F-expansion method; traveling wave solutions; conformable fractional derivative; Hirota-Satsuma equations.
Mathematics Subject Classifications Primary: 35C07, 49K15, 35R11, 35K99.

1. Introduction:

The investigation of the travelling wave solutions for nonlinear evolution equations (NLEEs) plays asignificantrole in mathematical and physical phenomena. Nonlinear wave phenomena appear in various scientific and engineering fields such as optical fibers, biology, solid state physics, chemical physics and geochemistry. With the perfection of symbolic computation software like Maple, many powerful and effective methods to find analytical and numerical solutions of nonlinear equations still have drawn the various attentions by different group of investigators.

Many powerful methods for scorching for exact solutions of non-linear evolution equations (NLEEs) have been established, such as Hirota's bilinear method [1], Backlund transformation [2], Homogeneous balance method [3],Burger's KdVmethod [12], Enhanced $\left(\frac{G'}{G}\right)$-expansion method[4, 5], $\left(\frac{G'}{G},\frac{1}{G}\right)$-expansion method [9],the exp(-Φ(ξ))-expansion method [6],modified generalized Riccati equation method[7],the F-expansion method [8], the improved F-expansion method [11, 18], conformable derivatives [13], the tan $(F(\xi)/2)$-expansion method [16], for the traveling wave solutions of the variant Bussinesq equations [10, 19, 20, 21], the modified Hirota-Satsuma method [17] and so on.

Many researchers utilized the fractional derivative [14, 15] and other well-known fractional derivatives alongside some analytical methods to solve many solitary wave problems such as the Hirota-Satsuma equations [13].

$$\begin{aligned} u_t + u_{xx} + 6uu_{xx} - 6vv_x = 0 \\ v_t - 2v_{xxx} - 6uv_x = 0 \end{aligned} \tag{1}$$

In this work, our targetis establishing some new and wide applicable solitary wave solutions to the nonlinear Hirota Satsuma equation through the operative method, called the improved F-expansion method with Riccati equation [22].The fresh analytical solutions achieved in this work are considered by hyperbolic functions, trigonometric functions and rational function.

The Maple software would be used in the solution of the system of algebraic equations obtained and also in the graphical illustrations of the solution.

The procedure of this article is as follows: In the Sect. 2, we give description of conformable derivative. In Sect. 3, we expound the improved F-expansion method. In Sect. 4, we apply this method to the Hirota-Satsuma equationsare formulated through the F-expansion method. In Sect. 5, we talk over the graphical illustration of some obtained solutions. In Sect. 6, we make a summary to the results that have been achieved.

2. Prefaces of conformable derivative:

Definition 1.1: (Conformable derivative operator).

Let $\alpha \in [0,1]$ A differential operatorD^α is conformable if and only if $D_t^0$is the identity operator and$D_t^1$is the classical differential operator. Specifically,$D_t^\alpha$is conformable if and only if for a differentiable function $u =$

$u(t), D_t^0 u(t) = u(t)$and $D_t^1 u(t) = \frac{d}{dt} u(t) = u^{'}(t)$.Note that under this definition the operator given via (ii) is not conformable.

Theorem 1.1: Let $\alpha \in (0,1)$and suppose $u(t)$and$v(t)$are α −differentiable at $t > 0$.
Then

a) $D_t^\alpha (t^p) = pt^{p-\alpha}$, for all $p \in \Re$
b) $D_t^\alpha (q) = 0$, for all constant function $u(t) = q$.
c) $D_t^\alpha (qu(t)) = qD_t^\alpha (u(t))$, for all q constant.
d) $D_t^\alpha (qu(t) + rv(t)) = qD_t^\alpha u(t) + rD_t^\alpha v(t)$, for all$q, r \in \Re$.
e) $D_t^\alpha \left(\frac{u(t)}{v(t)}\right) = \frac{v(t)D_t^\alpha u(t) - u(t)D_t^\alpha v(t)}{v^2(t)}$, $v(t) \neq 0$.
f) If, in addition to$u(t)$differentiable, then$D_t^\alpha u(t) = t^{1-\alpha} \frac{du}{dt}$.

Definition1.2: (Conformable derivative).
There are many exact hyperbolic function solutions for the time fractional variant bussinesq equations are created using the modified extended $tanh$ expansion method in the sense of the newly devised fractional derivative called the conformable fractional derivative by [12].
Let $u: [0, \infty) \to \mathbb{R}$ be a function. The α' s order conformable derivative of u is defined by

$$D_t^\alpha u(t) = \lim_{\varepsilon \to 0} \frac{u(t + \varepsilon t^{1-\alpha}) - u(t)}{\varepsilon} qD_t^\alpha (u(t)) \quad (2)$$

for all $t > 0$ and $\alpha \in (0,1)$. See also [13].

Theorem 1.2: Let $\alpha \in (0,1)$ such$u(t)$is differentiable and also α −differentiable
Let $v(t)$be a function defined in the range of$u(t)$also differentiable, then

$$D_t^\alpha (u(t) \circ v(t)) = t^{1-\alpha} v^{'}(t)(u^{'}(v(t)).$$ See also [14].

3. Explanation of the improved F-expansion method:

In this section, we will give the description of the improved F-expansion method to obtain the exact solutions of NLEEs. Now we consider the following non-linear fractional differential equation of the form:

$$P(u, D_t^\alpha u, D_x^\alpha u, D_{tt}^{2\alpha} u, D_{xx}^{2\alpha} u, D_t^\alpha D_t^\alpha u, \ldots\ldots) = 0; \quad 0 < \alpha < 1 \quad (3)$$

where α is order of the derivative of the function $u = u(x,t)$, P is a polynomial of $u(x,t)$ and its partial derivatives. Also, we use the traveling wave variables,

$$u(x,t) = u(\xi) \text{and} \xi = ax + b\frac{t^\alpha}{\alpha}, \quad (4)$$

Where a and $b$are non-zero constants. The traveling wave transformation equation (4) transforms equation (3) into an ordinary differentiation equation (ODE) for $u = u(x,t)$:

$$Q(u, u^{'}, u^{''} \ldots\ldots\ldots\ldots\ldots) = 0 \quad (5)$$

Where, $'$ is a derivative with respect to ξ.
Suppose the solution of equation (5) can be expressed as follows:

$$u(x,t) = u(\xi) = a_0 + \sum_{i=1}^m \left(a_i F^i(\xi) + \frac{b_i}{F^i(\xi)}\right) \quad (6)$$

where$a_0, a_i, b_i, (i = 1, 2, \ldots\ldots\ldots m)$ are arbitrary constants to be determined later; where$m$ is a positive integer determined by balancing the highest order derivative with the highest non-linear terms in the equation and F(ξ) be satisfy the Riccati differential equation:

$$F^{'}(\xi) = r + F^2(\xi) \quad (7)$$

where$r$is a constant. To established general solutions of the Riccati equation (7), we select the following three cases:

Case-I: When$r < 0$,the general hyperbolic solutions of (7) are:

$$F(\xi) = -\sqrt{-r}\tanh(\sqrt{-r}\,\xi)$$

$$F(\xi) = -\sqrt{-r}\coth(\sqrt{-r}\,\xi).$$

Case-II: When$r > 0$,the general trigonometric solutions of (7) are:

$$F(\xi) = \sqrt{r}\tan(\sqrt{r}\,\xi)$$

$$F(\xi) = -\sqrt{r}\cot(\sqrt{r}\,\xi).$$

Case-III: When $r = 0$, then the normal solution is:

$$F(\xi) = -\frac{1}{\xi}.$$

Therefore, putting equation (6) and its essential derivatives into equation(5) gives a polynomial in $F(\xi)$,collect all terms with the same power of$F(\xi)$together. Equating thecoefficients of the polynomial to zero, we will get a set of over-determine algebraic equations for $a_0, a_i, b_i (i = 1, 2, \ldots\ldots.. m)$, and r with the help of symbolic computation using Maple.

Finally, solving the algebraic equations and above possible solutions of Riccati equation into (5), we find the solution of equation (3).

4. Applications:

We consider the conformable time fractional Hirota-Satsuma equations version of (1) the form

$$\begin{cases} D_t^\alpha u + u_{xxx} + 6uu_x - 6vv_x = 0 \\ D_t^\alpha v - 2v_{xxx} - 6uv_x = 0 \end{cases} \tag{8}$$

Suppose that $u(x,t) = u(\xi)$ and $\xi = ax + b\frac{t^\alpha}{\alpha}$, (9)

The travelling wave variable equation (9) permits one convertingequation(8)into ODEs for $u = u(\xi)$ and $v = v(\xi)$ as follows:

$$\begin{cases} b\,u' + u''' + uu' - 6avv' = 0 \\ b\,v' - 2a^3v''' - 6auv' = 0 \end{cases} \tag{10}$$

Considering the homogeneous balancing betweenu''' $and\ vv'$, and between v''' $and\ uv'$ in Eq. (10) $m_1 + 3 = 2m_2 + 1$ and $m_2 + 3 = m_1 + m_2 + 1$ respectively.
So $m_1 = 2, m_2 = 2$.
Thus the Eq.(8) has a solution of the form:

$$\begin{cases} u(\xi) = a_0 + a_1F(\xi) + a_2F^2(\xi) + \frac{b_1}{F(\xi)} + \frac{b_2}{F^2(\xi)} \\ v(\xi) = c_0 + c_1F(\xi) + c_2F^2(\xi) + \frac{d_1}{F(\xi)} + \frac{d_2}{F^2(\xi)} \end{cases} \tag{11}$$

where from equation (7)

$$F'(\xi) = r + F^2(\xi) \text{and} F''(\xi) = 2(F(\xi)r + F^2(\xi)) \tag{12}$$

Now putting the values of Eq. (11) and their necessary derivatives along with Eq.(12) into Eq.(8),yields a set of algebraic equations for $a_0, a_1, a_2, b_1, b_2, c_0, c_1, c_2, d_1, d_2, a, b, r$. These algebraic equations system are obtained as

$-48a^3c_2 - 12aa_2c_2 = 0$
$-6aa_2c_1 - 12a^3c_1 - 12aa_1c_2 = 0$
$-80a^3c_2r - 12raa_2c_2 - 6aa_1c_1 - 12aa_0c_2 + 2bc_2 = 0$
$-6raa_2c_1 - 16a^3c_1r - 12raa_1c_2 - 12ab_1c_2 + 6aa_2d_1 + bc_1 - 6aa_0c_1 = 0$
$-32a^3c_2r^2 - 6raa_1c_1 - 12raa_0c_2 + 2bc_2r - 6ab_1c_1 + 12aa_2d_2 + 6aa_1d_1 - 12ab_2c_2 = 0$
$-12rab_1c_2 + 6raa_2d_1 - 6raa_0c_1 + 6aa_0d_1 - 6ab_2c_1 + 12aa_1d_2 + bc_1r + 4a^3d_1r - 4a^3c_1r^2 - bd_1 = 0$
$-6rab_1c_1 + 12raa_2d_2 + 6raa_1d_1 - 12rab_2c_2 - 2bd_2 + 6ab_1d_1 + 32a^3d_2r + 12aa_0d_2 = 0$
$6raa_0d_1 - 6rab_2c_1 + 16a^3d_1r^2 + 12raa_1d_2 - bd_1r + 6ab_2d_1 + 12ab_1d_2 = 0$
$-2bd_2r + 6rab_1d_1 + 80a^3d_2r^2 + 12raa_0d_2 + 12ab_2d_2 = 0$
$6rab_2d_1 + 12rab_1d_2 + 12a^3d_1r^3 = 0$
$48a^3d_2r^3 + 12rad_2b_2 = 0$

By using Maple solving the above system, we obtain

Case 1

$a_0 = -\frac{16ra^3 - b}{6a}, a_1 = 0, a_2 = -4a^2, b_1 = 0, b_2 = -4a^2r^2,$
$c_0 = \pm\frac{(-8ra + 16ra^3 - b - ab)\left(\sqrt{4a^3 - 2a}\right)}{6a(2a^2 - 1)}, c_1 = 0, c_2 = \pm 2\sqrt{4a^3 - 2a}, d_1 = 0,$
$d_2 = \pm 2\left(\sqrt{4a^3 - 2a}\right)r^2$

Thus the solution is:

$$u_1(x,t) = -\frac{16ra^3 - b}{6a} + 4a^2r\,tanh^2\left(\sqrt{-r\xi}\right) + \frac{4a^2r}{tan\,h^2\left(\sqrt{-r\xi}\right)} \tag{13}$$

$$v_1(x,t) = \pm\frac{(-8ra + 16ra^3 - b - ab)\left(\sqrt{4a^3 - 2a}\right)}{6a(2a^2 - 1)} \pm 2\left(\sqrt{4a^3 - 2a}\right)r\,tanh^2\left(\sqrt{-r\xi}\right) \pm \frac{2\left(\sqrt{4a^3 - 2a}\right)r}{tan\,h^2\left(\sqrt{-r\xi}\right)} \tag{14}$$

Case 2

$a_0 = -\dfrac{16ra^3 - b}{6a}, a_1 = 0, a_2 = 0, b_1 = 0, b_2 = -4a^2r^2,$

$c_0 = \pm\dfrac{(-8ra + 16ra^3 - b - ba)}{3\sqrt{4a^3 - 2a}},$

$c_1 = 0, c_2 = 0, d_1 = 0, d_2 = \pm 2\left(\sqrt{4a^3 - 2a}\right)r^2$

Therefore, the solution is:

$$u_2(x,t) = -\frac{16ra^3-b}{6a} + \frac{4a^2r}{tan\,h^2(\sqrt{-r\xi})} \quad (15)$$

$$v_2(x,t) = \pm\frac{(-8ra+16ra^3-b-ba)}{3\sqrt{4a^3-2a}} \pm \frac{2\left(\sqrt{4a^3-2a}\right)r}{tan\,h^2(\sqrt{-r\xi})} \quad (16)$$

Case 3

$a_0 = -\dfrac{(16ra^3 - b)}{6a},\ a_1 = 0, a_2 = 0, b_1 = 0, b_2 = -4a^2r^2, c_0 = c_0, c_1 = 0, c_2 = 0, \qquad d_1 = 0, d_2 = d_2$

Also, the solution is:

$$u_3(x,t) = -\frac{16ra^3-b}{6a} - \frac{4a^2r}{tan\,h^2(\sqrt{-r\xi})} \quad (17)$$

$$v_3(x,t) = c_0 + \frac{d_2}{tan\,h^2(\sqrt{-r\xi})} \quad (18)$$

The above Cases 1-3, we use $r < 0$ to get those solutions which containing hyperbolic functions from Equation (7); but if we are taken $r > 0$ expected to change to trigonometric functions as explained in sect. 3. Here the value of ξ in the cases 1-3 is given by

$$\xi = ax + b\frac{t^\alpha}{\alpha}. \quad (21)$$

5. Graphical representations:

Graphical illustration is an important tool for communication and it represents clearly the solutions of the problems. We give some graphical representations of the conformable time-fractional on Hirota-Satsuma equations at different time levels through different values of α: $u(x,t)$ and $v(x,t)$ are given in following figure 1-6. Figure 1 and Figure 4 are plotted for the solutions of equations(13) and (14) for the values a = 2, b = -1, α = 1.0, r = 5 within the interval $0 \le x \le 5$ and $10 \le t \le 5$. The shape of the solutions are singular Soliton. Figure 2 and Figure 5 are plotted for the solutions of equations.(15) and (16) for the values a = 1, b = -2, α = 0.5, r = -5 within the interval $0 \le x \le 40$ and $10 \le t \le 50$. The shape of the solutions are single Soliton.

Profiles of $u(x,t)$ at different time levels:

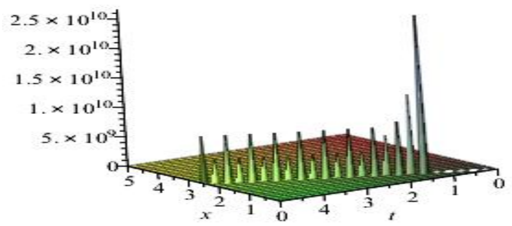

Fig. 1 Singular Soliton of Eq. (13) for a = 2, b = -1, α = 1.0, r = 5 within the interval $0 \le x \le 5$ and $10 \le t \le 5$.

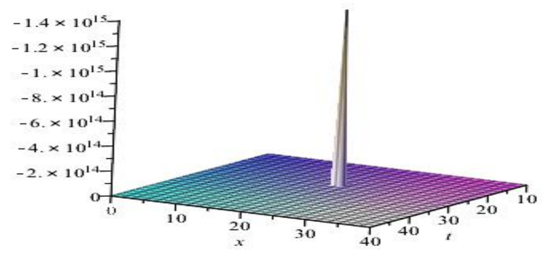

Fig. 2 Single Soliton of Eq. (15) for a = 1, b = -2, α = 0.5, r = -5 within the interval $0 \le x \le 40$ and $10 \le t \le 50$.

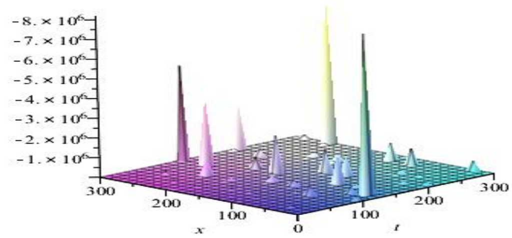

Fig. 3 Singular Soliton of Eq. (17) for a = 2, b = 2, α = 0.5, r = 3 within the interval $0 \leq x \leq 300$ and $0 \leq t \leq 300$.

Profiles of $v(x,t)$ at different time levels:

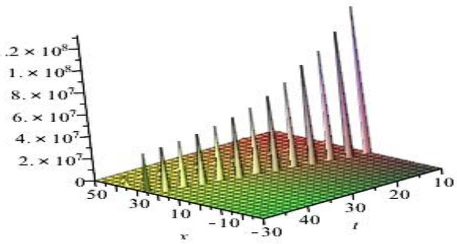

Fig. 4 Singular Soliton of Eq. (14) for a = 2, b = -1, α = 1.0, r = 5 within the interval $-50 \leq x \leq 50 and -50 \leq t \leq 50$.

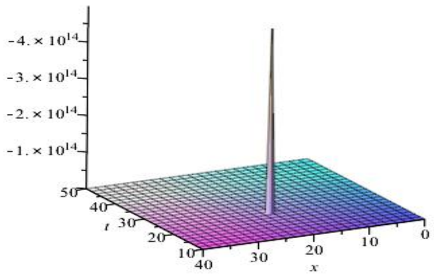

Fig. 5 Single Soliton of Eq. (16) for a = 1, b = -2, α = 0.5, r = -5 within the interval $0 \leq x \leq 40$ and $10 \leq t \leq 50$.

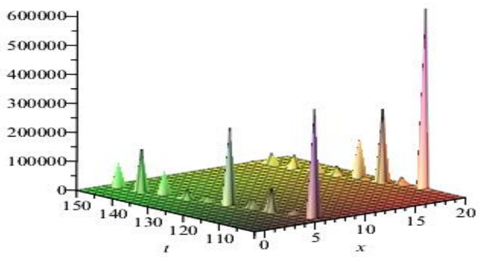

Fig. 6 Periodic Soliton of Eq. (18) for a = 2, b = 2, α = 0.5, r = 3 within the interval $-1 \leq x \leq 20$ and $100 \leq t \leq 150$.

For the values a = 2, b = 2, α = 0.5, r = 3, the equations (17) and (18) has been plotted within the interval $0 \leq x \leq 300$ and $0 \leq t \leq 300$ in Figure 3 and Figure 6. Both of the shapes of the solutions are singular Soliton and Periodic Soliton respectively.

6. Conclusion

In this present work, the improved F-expansion method combined with Riccati equation has been applied effectively to launch abundant exact traveling wave solutions for the Hirota-Satsuma equations. We have successfully achieved exact traveling wave solutions with three classes of explicit solutions-hyperbolic, trigonometric and rational solutions with some free parameters. These exact solutions have been verified by symbolical computation system-Maple. Lastly, it is worthwhile to indication that the projected method is also a straightforward, short, encouraging and powerful method for other nonlinear evolution equations in mathematical physics.

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