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ZERO-DIVISOR GRAPHS OF UPPER TRIANGULAR MATRICES OVER FINITE FIELDS

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Abstract: In this paper we construct a method for computing an upper bound for the number of matrix multiplications needed to construct the zero-divisor graph of a ring of upper triangular matrices over a finite field. We compare this method to the brute force method.
Keywords: Zero-divisor graphs, upper triangular matrices.

1. Introduction

Let R be a finite commutative ring, $1 \neq 0$. Let $Z^*(R)$ denote the set of non-zero zero-divisors of R. The zero-divisor graph of R, denoted $\Gamma(R)$, is the undirected graph whose vertices are labeled by the elements of $Z^*(R)$. There is an edge in $\Gamma(R)$ between the vertices r and s if and only if $rs = 0$. In this case we say that r and s are adjacent.

Beck [6] first defined zero-divisor graphs for commutative rings in the context of coloring of graphs. Papers by Anderson and Naseer [5] and Anderson and Livingston [4] followed. In the last several years there has been a large number of papers on this topic; see Anderson, Frazier, and Livingston [3], Anderson and Badawi [2], and Coykendall, Sather-Wagstaff, Sheppardson, and Spiroff [9] for surveys and extensive bibliographies.

Redmond [13, 14] introduced the concept of the zero-divisor graph for a non-commutative ring R. In this case $\Gamma(R)$ is a directed graph. If $r, s \in Z^*(R)$ and $rs = 0$, then there is a directed edge from r to s. Bozic and Petrovic [7] and Akbari and Mohammadian [1] studied the zero-divisor graphs of matrix rings. Li [12] and Li and Tucci [11] studied the zero-divisor graphs of upper triangular matrix rings.

Using the language of graph theory, we denote the set of vertices of $\Gamma(R)$ as $V(\Gamma(R)) = Z^*(R)$ and the set of edges of $\Gamma(R)$ as $E(\Gamma(R)) = \{(r,s) \mid r, s \in Z^*(R) \text{ and } rs = 0\}$. For a general background on graph theory, see Chartrand, Lesniak, and Chang [8].

In the next section, we include some preliminary results about upper triangular matrices. In section three, we construct an upper bound for the number of matrix multiplications required to calculate zero-divisor graphs. Finally, in the fourth section, we compare the savings of the optimization to the brute force approach.

In what follows we denote the ring of $n \times n$ upper triangular matrices over a ring R by $T_n(R)$. We denote the set of zero-divisors in this ring by $Z^*(T_n(R))$ or simply by Z^* if the context is clear.

2. Preliminary Results

In this section, we include some basic results about upper triangular matrices for the sake of completeness. Let R be a finite ring. The following result is well known.

Proposition 2.1

Every element of a finite ring is either a zero-divisor or a unit.

Proposition 2.2

Let $M \in T_n(R)$.

(1) The matrix M is a unit iff m_{jj} is a unit for all $1 \leq j \leq n$.

(2) The matrix M is a zero-divisor iff m_{jj} is a zero-divisor for some $1 \leq j \leq n$.

(3) The matrix M is nilpotent iff m_{jj} is nilpotent for all $1 \leq j \leq n$.

Proof

Proofs (1) and (2) follow from [11, Them. 2.5].

For (3) Assume that $M^t = 0$ for some integer t. The elements on the diagonal of M^t are all of the form $m_{jj}^t = 0$ for $1 \leq j \leq n$. Hence m_{jj} is nilpotent for all $1 \leq j \leq n$.

Conversely, assume that the diagonal elements of M are all nilpotent. For some integer v the diagonal elements of M^v are all zero. Hence without loss of generality we can assume that all the diagonal elements of M are 0.

We show that M^2 has fewer non-zero columns than M. Let s be the column number of the first non-zero column in M^2. Let m_{rs} be a non-zero entry in this column. Then $m_{rs} = \sum_{\acute{s}} m_{r\acute{s}} m_{\acute{s}s}$ where $\acute{s} < s$. Hence the elements in column s in M^2 come from elements in column $\acute{s}$ in M, where $\acute{s} < s$. In particular the elements of column s of M^2 are all 0. Repeating this argument we see that for large enough t we have that $M^t = 0$. □

3. An Upper Bound for the Number of Matrix Multiplications to Construct $\Gamma(T_n(\mathbb{F}))$ for a Finite Field $\mathbb{F}$

Let $\mathbb{F}$ be a finite field of size f. In this section we compute an upper bound for the number of matrix multiplications which are needed to construct the zero-divisor graph for $T_n(\mathbb{F})$. We do this by constructing a graph FINAL which contains $T_n(\mathbb{F})$ as a subgraph. We construct the graph FINAL in three stages. First we construct a graph NILP whose vertices consist of the non-zero nilpotent matrices of $T_n(\mathbb{F})$. We then construct a graph NONNILP whose vertices consist of the non-nilpotent zero-divisors of $T_n(\mathbb{F})$. The graph FINAL is the graph direct product of NILP and NONNILP.

In this section, if G is any graph, we denote the vertices of a graph G by V_G and the edges of G by E_G. If S is any set, then $|S|$ denotes the size of S.

3.1. Nilpotent Matrices Graph

The *nilpotent matrices graph* NILP is the complete directed graph whose vertices consist of non-zero nilpotent matrices. By Proposition 2.2 these matrices are precisely the matrices of $T_n(\mathbb{F})$ whose diagonals consist entirely of 0's. Note that NILP is not necessarily a zero-divisor graph, since the product of nilpotent matrices is not necessarily 0.

Proposition 3.1

The number of vertices and edges in NILP is:

$$|V_{NILP}| = f^{\frac{n(n-1)}{2}} - 1$$

$$|E_{NILP}| = \left(f^{\frac{n(n-1)}{2}} - 1\right)\left(f^{\frac{n(n-1)}{2}} - 2\right)$$

3.2. Non-Nilpotent Matrices Graph

The *non-nilpotent matrices graph* NONNILP is a graph whose vertices consist of the non-nilpotent matrices of $T_n(\mathbb{F})$ which are zero-divisors. These are precisely the matrices with at least one 0 and one non-zero element on the diagonal. In order to construct this graph we use the graph constructed by LaGrange in [10].

Definition 3.2

The LaGrange graph LAG is the zero-divisor graph of:

$$\left(\prod_{1}^{n} \mathbb{Z}_2\right)$$

Proposition 3.3

$$|V_{LAG}| = 2^n - 2$$

$$|E_{LAG}| = \frac{3^n + 1}{2} - 2^n$$

Proof

By [10, Lemma 2.1] we have $|V_{LAG}| = 2^n - 2$ and

$$|E_{LAG}| = \sum_{j=2}^{n} (2^{j-1} - 1)\binom{n}{j}$$

We can simplify this latter formula as follows. Write:

$$\sum_{j=2}^{n}(2^{j-1}-1)\binom{n}{j}=\sum_{j=2}^{n}(2^{j-1})\binom{n}{j}-\sum_{j=2}^{n}\binom{n}{j}$$

Now

$$\sum_{j=2}^{n}2^{j-1}\binom{n}{j}=\frac{1}{2}\sum_{j=2}^{n}2^{j}\binom{n}{j}$$

$$=\frac{1}{2}\left[\sum_{j=0}^{n}2^{j}\binom{n}{j}-\binom{n}{0}-2\binom{n}{1}\right] \quad (1)$$

$$=\frac{1}{2}[3^{n}-1-n]$$

and

$$\sum_{j=2}^{n}\binom{n}{j}=\sum_{j=0}^{n}\binom{n}{j}-\binom{n}{0}-\binom{n}{1} \quad (2)$$

$$=2^{n}-1-n$$

Subtracting the expressions from equations (1) and (2) yields the result. □

Proposition 3.4

$$|V_{NONNILP}|=(2^{n}-2)\left(f^{\frac{n(n-1)}{2}}\right)$$

$$|E_{NONNILP}|=2f^{n(n-1)}\left[\frac{3^{n}+1}{2}-2^{n}\right]$$

Proof

Take the set of non-nilpotent zero-divisors in $T_n(\mathbb{F})$ and divide this set of matrices into disjoint subsets $A_1, A_2, \ldots$; two matrices are in the same subset precisely if they have 0's at the same location in their diagonals. Note that no two matrices in the same subset multiply to 0. Thus, each A_j is a totally disconnected graph.

For each A_j we construct a vector a_j as follows. Denote the m position of a by $a_j(m)$. Define $a_j(m)$ by

$$a(m)=\begin{cases}0, & if\ each\ matrix\ in\ A_j\ has\ 0\ in\ the\ (m,m)\ position\\ 1, & otherwise\end{cases}$$

Call a_j the *diagonal vector* of A_j. The zero-divisor graph for the diagonal vectors is precisely the Lagrange graph.

We now construct NONNILP. If $M_j \in A_j$ and $M_k \in A_k$, then a necessary condition for $M_jM_k = 0$ is that $a_j \cdot a_k = 0$. Therefore we replace each vertex a_j in the Lagrange graph by the graph A_j. Connect each matrix in A_j to each matrix in A_k precisely when $a_j \cdot a_k = 0$. Note that the resulting graph is directed, unlike the Lagrange graph.

To count the number of vertices in NONNILP, note that since the matrices in each A_j have identical diagonals, they can differ in $\frac{n(n-1)}{2}$ places. Hence A_j contains $f^{\frac{n(n-1)}{2}}$ matrices. Each A_j corresponds to a vertex in the LaGrange graph, and there are (2^n-2) vertices in the LaGrange graph. The result now follows. □

Theorem 3.5

$$|V_{FINAL}|=|V_{NILP}|+|V_{NONNILP}|$$

$$|E_{FINAL}|=|E_{NILP}|+|E_{NONNILP}|+2|V_{NILP}|\cdot|V_{NONNILP}|$$

Proof

The graph FINAL is the direct product of the graphs NILP and NONNILP. □

4. Comparison to Brute Force

In this section, we compare the algorithm in the previous section to the brute force algorithm. Note that the savings in our algorithm comes only from constructing NONNILP. The savings from algorithm above does not affect the construction of NILP or the graph direct product of NILP and NONNILP. To compare the algorithms, we first compute the number of matrix multiplications required by brute force.

Lemma 4.1

The number of non-zero non-nilpotent zero-divisors is:

$$f^{\frac{n(n-1)}{2}}(f^n - (f-1)^n - 1)$$

Proof

Let M be a non-nilpotent matrix, which is a zero-divisor. The number of elements above the diagonal is:

$$f^{\frac{n(n-1)}{2}}$$

The total number of possible diagonals is f^n. From this we subtract the number of diagonals which are all non-zero; this number is $(f-1)^n$. Finally, we subtract 1 to avoid including the zero diagonal. □

We denote the expression in Lemma 4.1 by X.

Corollary 4.2

The number of matrix multiplications in the brute force algorithm is $X(X-1)$.

Proof

We multiply all ordered pairs of distinct matrices. □

Proposition 4.3

The brute force algorithm is $O\left(n^2 f^{n^2+n-2}\right)$.

Proof

Since $X = f^{\frac{n(n-1)}{2}}(f^n - (f-1)^n - 1)$, then the leading term of $X^2 - X$ is the leading term of X^2, which is:

$$\begin{aligned} f^{n(n-1)}(nf^{n-1})^2 &= f^{n^2-n} n^2 f^{2n-2} \\ &= n^2 f^{n^2+n-2} \end{aligned}$$

□

Proposition 4.4

The algorithm using the LaGrange graph is $O\left(3^n f^{n^2-n}\right)$.

Proof

By Proposition 3.4 the number of matrix multiplications is $2f^{n(n-1)}\left[\frac{3^n+1}{2} - 2^n\right]$ where the leading term is a multiple of $3^n f^{n^2-n}$. □

Theorem 4.5

The algorithm using the LaGrange graph is more efficient than the brute force algorithm.

Proof

From Proposition 4.3 and Proposition 4.4, for $f > 2$ we have:

$$\begin{aligned} 3^n f^{n(n-1)} &\leq f^n f^{n(n-1)} \\ &= f^{n^2} \\ &< n^2 f^{n^2+n-2} \end{aligned}$$

□

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