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Between $\mathcal{K}$C-Regularity & $\mathcal{K}$-Regularity of Topological Spaces

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Abstract: The present paper consists of the generalization of the concepts of regular spaces in terms of nearly open sets viz. p-open, s-open, α-open, b-open & β-open sets. We introduce such spaces in abbreviation as $\mathcal{K}$C-regular, $\mathcal{K}$-regular & $\mathcal{K}$*-regular spaces where $\mathcal{K}$ = p,s,α,b&β and highlight the related characterization & preservation criteria of these weaker form of spaces along with the focus on some basic properties like " Every $\mathcal{K}$*-regular and $\mathcal{K}$-T_1 space is $\mathcal{K}$-T_2 space". And, also, on the basis of research, some factual observations have been established.

The paper, also, deals with the weaker form named by $\mathcal{K}$C-regular space.

Key words: $\mathcal{K}$-regular , $\mathcal{K}$*-regular, $\mathcal{K}$C-regular, $\mathcal{K}$-open, $\mathcal{K}$-closed sets ($\mathcal{K}$ = p,s,α,b &β).

Introduction:

1982 is the year for projection and extensive research of the concepts of pre-open sets along with pre-continuous & pre-irresolute mapping in general topology by A.S. Mashour et. al.[3].C.Kuratowski mentioned that a set in a space is said to be regular open set or an open domain if it is the interior of its closure.

As a recall, in 1963 a semi-open set was defined as a subset A of the space (X,T) for which there exists O ∈ T such that O ⊂A ⊂ Cl(O)[4].

In 1965 α-sets were introduced in the manner that a subset A of a space(X,T) is an α - open set iff A ⊂ int (cl(int(A)))[5].

In 1981, pre-open sets were introduced, having the criteria that a subset A of a space(X,T) is known to be pre- open set iff A⊂ int(cl(A))[2].

The semi-pre open set, called by D. Andrijewic was conceptualized under the name β- open set by M.E. Abd El-Monsef etc in 1983 as a subset A of a space (X,T) satisfying
A ⊂ cl (int(cl(A)))[7].

Anderijevic introduced a new class of generalized open sets in a topological space, the so called b-open sets. The b-open sets is defined as a subset A of a space (X,T) such that
A⊂ {cl(int(A))} ∪{int(cl(A))}[10].

The class of b-open sets in topological space is contained in the class of semi –pre- open sets and contains all semi-open sets & pre -open sets.

Also, the class of b-open sets generates the same topology as a class of pre- open sets [10].

These nearly open [5] sets are used to generalize regular spaces introduced in abbreviation as $\mathcal{K}$-regular, $\mathcal{K}$*-regular and $\mathcal{K}$C-regular spaces where $\mathcal{K}$ = p, s, α, b & β.

§1.$\mathcal{K}$C-regular, $\mathcal{K}$*-regular & $\mathcal{K}$-regular spaces:

The concept of a regular space is generalized in terms of $\mathcal{K}$-open sets where $\mathcal{K}$ = p,s,α,b &β.

Definition (1.1):

A topological space (X,T) is said to be $\mathcal{K}$–closed regular(briefly, $\mathcal{K}$C-regular) space if for every $\mathcal{K}$-closed set F and each point x ∉ F, there exist disjoint open sets U & V such that F ⊂ U, x∈ V, and U ∩ V = φ.

Definition (1.2):

A topological space (X,T) is said to be $\mathcal{K}$*-regular space if for every $\mathcal{K}$-closed set F and each point x ∉ F, there exist $\mathcal{K}$-open sets U & V such that F ⊂ U, x∈ V, and U ∩ V = φ.

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Definition (1.3):

A topological space (X,T) is said to be $\mathcal{K}$-regular space if for every closed set F and each point x ∉ F, there exist $\mathcal{K}$-open sets U & V such that F ⊂ U, x∈ V and U ∩ V = φ.

The following examples indicate that the above concepts are different:

Example (1.4):

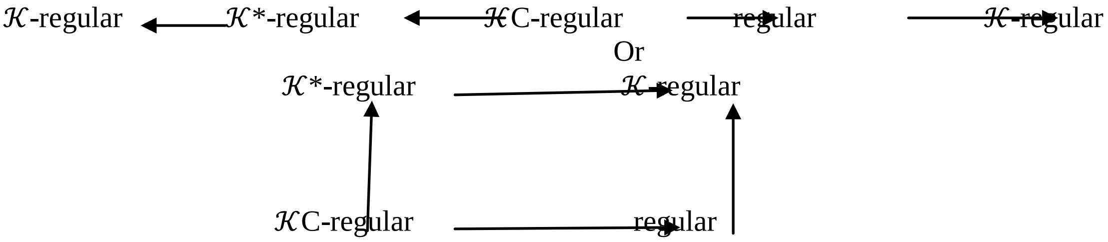
(a) Let X = {a,b,c,d} and T = {φ, {b},{c},{b,c}, X}, then for the topological space (X,T),T^c = The class of all closed subsets = {φ,{a,d},{a,b,d},{b,c,d}, X},

And SO(X) = the class of all Semi-open sets ={φ,{b},{c},{a,b},{a,c},{b,c},{b,d},{c,d},{a,b,c},{a,b,d},{a,c,d},{,b,c,d},X}.
i.e. ℘(X) - { {a},{d},{a,d}}.
SC(X) = the class of all semi-closed sets = ℘(X) - { {b,c},{a,b,c},{b,c,d}}.
Hence, (X,T) is s-regular space and it is neither regular nor S*-regular(i.e. semi-regular in the literature) space. Also, (X,T) is not SC-regular.

(b) Consider the topology T = { X, φ, {a}, {b,c}} on the ground set X= {a,b,c}.
Then PO(X,T) = the class of all pre-open sets = {{a},{b},{c},{a,b},{b,c},{a,c}, X, φ}.
& PC(X) ={φ,{a},{b},{c},{a,b},{b,c},{a,c},X}. Thus, (X,T) is pre-regular. On the other hand(X,T) is, also, p*- regular but not PC-regular.

(c) Let X = {a,b,c,d} and
T = {φ, {a,d},{a,b,d},{a.c,d}, X} so that T^c = the class of all closed sets = {φ,{b,c},{c},{b},X}.
Now, PO(X) = ℘(X) - { {b},{c},{b,c}}& PC(X) =℘(X) - { {a,d},{a,b,d},{a,c,d}}.
Therefore, (X,T) is neither regular space nor pre-closed regular space. But p- regular space.

The implication diagram is given as:



Factual observations:

(a) Every regular space is 𝒦 -regular.
(b) Every $\mathcal{K}^*$ -regular space is 𝒦 -regular.
(c) Every 𝒦- regular space need not be a T_1 space e.g. in the example (1.4)(b), the space (X,T) is pre-regular but not a T_1 – space.

Characterization theorem (1.5):

A topological space (X,T) is characterized to be 𝒦*-regular (resp. 𝒦C-regular) iff for every x∈ X and for every 𝒦 - open set U containing x there exists a 𝒦 –open(resp. open) set V such that x∈ V ⊆ 𝒦cl (V)(resp. cl(V)⊆ U where 𝒦 = p, s, α, b & β.

Proof:

Let (X,T) be a 𝒦*-regular (resp. 𝒦C-regular) space. Let x∈ X and U be a 𝒦 –open(resp. open) set containing x . Then U^c is 𝒦-closed(resp. closed) and x ∉ U^c. Since X is 𝒦*-regular (resp. 𝒦C-regular), there exist a 𝒦-open(resp. open) sets V &W such that x∈ V, U^c ⊂ W and V ∩ W = φ. And this means that x∈ V, W^c ⊂ U. Since, W^c is 𝒦-closed (resp.closed), hence, we have x∈ V ⊆ 𝒦cl (V)(resp. cl(V) ⊆ W^c ⊆ U.

Conversely, suppose that the given condition holds good. Let F be 𝒦 –closed(resp. closed) and x ∉ F. Then F^c is 𝒦 –open(resp. open) and x ∈ F^c. So, according to the given condition there exists a 𝒦 –open(resp. open) set V such that x∈ V ⊆ 𝒦cl (V)(resp. cl(V)⊆ F^c.

Now, $(\mathcal{K}cl\ (V))^C$(resp. $(cl(V))^C$ is a 𝒦 –open(resp. open) set such that x∈ V, F ⊆ $(\mathcal{K}cl\ (V))^C$(resp. $(cl(V))^C$ and V ∩ $(\mathcal{K}cl\ (V))^C$(resp. $(cl(V))^C$ = φ. This establishes that (X,T) is a 𝒦*-regular (resp. 𝒦C-regular) space. Hence, the theorem.

Theorem (1.6):

A topological space (X,T) is 𝒦 -regular iff for every x ∈ X and for every open set U containing x there exists a 𝒦 -open set V such that x∈ V ⊆ 𝒦cl V⊆ U where 𝒦 = p, s, α, b & β.

Proof: The result follows in the similar manner with proper changes according to the context in theorem (1.5).

Theorem (1.7):

A topological space (X,T) is 𝒦C-regular space if for every 𝒦-closed set F in (X,T) and each point x ∉ F, there exist open sets U & V in (X,T) such that x∈ U, F ⊂ V and cl(U) ∩ cl(V) = φ where 𝒦 = p, s, α, b & β.

Proof :

Suppose that F is a 𝒦-closed set in a 𝒦C-regular space(X,T) such that x ∉ F where 𝒦 = p, s, α, b & β. By the concept, there must exist open sets U_0 &V in (X,T) such that x∈ U, F ⊂ V and U_0 ∩ V = φ. Now, U_0 ∩ V = φ ⇒ U_0 ∩ cl(V) = φ.

Since, cl(V) is closed, it is 𝒦-closed. Also, by assumption x ∉ F& F⊂V i.e. x ∉ F& F⊂V ⊂cl(V) i.e. x ∉cl(V). Thus, 𝒦C-regularity of (X,T) provides that there exist open sets G & H of (X,T) such that x∈ G, cl(V)⊂ H and G ∩ H = φ. This implies that cl(G) ∩ H = φ.

Next, let U = U_0 ∩G, then U is an open set. Obviously, x∈ U. Consequently, we get that for a 𝒦-closed set F and x ∉ F, there exist open sets U & V in (X,T) such that x∈ U,
F ⊂ V and cl(U) ∩ cl(V) = φ .

The converse part is trivial. Hence, the theorem.

Preservation Criteria :

Before the discourse on preservation criteria for $\mathcal{K}$C-regularity & $\mathcal{K}$*-regularity of a topological space (X,T), we cite the following definitions of $\mathcal{K}$-homeomorphism, weakly continuous mapping, $\mathcal{K}$-open & $\mathcal{K}$-closed mappings alongwith $\mathcal{K}$-topological property where $\mathcal{K}$ = p, s, α, b & β.

Definition (1.8)($\mathcal{K}$-homeomorphism)**:** ($\mathcal{K}$ = p, s, α, b & β).

A bijection f: (X,T)→(Y,σ) from one topological space(X,T) to another topological space(Y,σ) is called $\mathcal{K}$ -homeomorphism if f is both $\mathcal{K}$ -irresolute and $\mathcal{K}$ -open mapping where $\mathcal{K}$ = p, s, α, b & β.

Or

A one-one mapping f: (X,T)→(Y,σ) from one topological space(X,T) to another topological space(Y,σ) is called $\mathcal{K}$ -homeomorphism iff images f(U) of sets U ∈ $\mathcal{K}$O(X,T) are $\mathcal{K}$ -open sets in (Y,σ) and inverse images $f^{-1}(V)$ of sets V ∈ $\mathcal{K}$O(Y,σ) are $\mathcal{K}$ -open sets in (X,T), where $\mathcal{K}$ = p, s, α, b & β.

Definition (1.9): Weakly Continuous mapping:

A mapping f: (X,T)→(Y,σ) from one topological space(X,T) to another topological space(Y,σ) is weakly continuous mapping iff for each point x∈ X and each open set V in (Y,σ) containing f(x), there exists an open set U containing x such that f(U)⊂cl(V).

Definition (1.10): $\mathcal{K}$-topological property($\mathcal{K}$ = p, s, □, b & □):

A property of topological spaces preserved by $\mathcal{K}$-homeomorphisms is called a $\mathcal{K}$ -topological property.

Definition (1.11): $\mathcal{K}$-open/ $\mathcal{K}$-closed mapping ($\mathcal{K}$ = p, s, □, b & □):

A mapping f: (X,T)→(Y,σ) from one topological space(X,T) to another topological space(Y,σ) is said to be $\mathcal{K}$-open(resp. $\mathcal{K}$-closed) mapping if for every open set U(resp. closed set V) in (X,T), f(U) is $\mathcal{K}$-open (resp. $\mathcal{K}$ -closed) in (Y,σ).

Theorem (1.12):

If the mapping f:(X,T) → (Y,σ) is weakly-continuous & $\mathcal{K}$-closed, injection and (Y,σ) is $\mathcal{K}$C -regular then (X,T) is regular where $\mathcal{K}$ = p, s, α, b & β.

Proof:

Let F be any closed set in (X,T) and x ∉ F. Since, f is $\mathcal{K}$-closed, f(F) is $\mathcal{K}$-closed in (Y,σ), and f(x) ∉ f(F). Since, (Y,σ) is $\mathcal{K}$C-regular, by the theorem (1.7) there exist open sets U & V in (X,T) such that x∈ U, F ⊂ V and cl(U) ∩ cl(V) = φ .

Since, f is weakly continuous it follows that x ∈ $f^{-1}(U) \subseteq int(f^{-1}(cl(U)))$,F ⊆ $f^{-1}(V) \subseteq int(f^{-1}(cl(V)))$ and $int(f^{-1}(cl(U))) \cap int(f^{-1}(cl(V))) = \varphi$.

Consequently, (X,T) is a regular space.

Hence, the theorem.

Theorem (1.13):

If f:(X,T) → (Y,σ) is a bijection, $\mathcal{K}$-continuous & $\mathcal{K}$-open mapping with domain (X,T) as a $\mathcal{K}$C -regular space then (Y,σ) is a $\mathcal{K}$ -regular space where $\mathcal{K}$ = p, s, α, b & β.

Proof:

Let y ∈ Y and F be any closed set in Y such that y ∉F. Since, the mapping f is bijection, there exists a point x ∈ X such that f(x) = y ⇒ x = $f^{-1}(y) \notin f^{-1}(F)$. Since, (X,T) is a $\mathcal{K}$C -regular space, there exist open sets M, N in X such that x ∈ M, $f^{-1}(F)$ ⊂ N and M ∩N = ϕ. Since, f is $\mathcal{K}$-open map, f(M), f(N) are $\mathcal{K}$ -open sets in (Y,σ) . Now, we have x ∈ M ⇒f(x) ∈ f(M) ⇒ y ∈ f(M); $f^{-1}(F)$⊂ N ⇒ $f[f^{-1}(F)]$ ⊂ f(N) ⇒ F ⊂f(N) and M ∩ N = ϕ provides that f(M) ∩ f(N) = f(ϕ) = ϕ, since f is a bijection. Thus, for every point y ∈ Y and each closed set F in (Y,σ) such that y ∉F, there exist $\mathcal{K}$ -open sets f(M), f(N) in (Y,σ) such that y ∈ f(M), F⊂ f(N) and f(M) ∩ f(N) = ϕ. Consequently, (Y,σ) is a $\mathcal{K}$ -regular space.

Hence, the theorem.

Theorem (1.14):

If f : (X,T) → (Y,σ) is a bijection, $\mathcal{K}$-homeomorphism and (X,T) is $\mathcal{K}$*- regular space then (Y,σ) is also a $\mathcal{K}$* -regular space where $\mathcal{K}$ = p, s, α, b & β.

Proof:

Let y ∈ Y and F is any $\mathcal{K}$ -closed set in (Y,σ) such that y ∉ F. Since f is bijection, there exists a point x ∈ X such that f(x) = y ⇒ x = $f^{-1}(y)$. Again, since f is $\mathcal{K}$- continuous, hence, $f^{-1}(F)$ is a $\mathcal{K}$ -closed set in (X,T). Also, y ∉ F ⇒ $f^{-1}(y) \notin f^{-1}(F)$ i.e. x ∉ $f^{-1}(F)$.Since (X,T) is $\mathcal{K}$ *-regular space, there exist $\mathcal{K}$ -open sets M, N in (X,T) such that x ∈ M, $f^{-1}(F)$ ⊂ N and M ∩N = ϕ. Since f is $\mathcal{K}$ –open map, f(M), f(N) are $\mathcal{K}$ - open sets in (Y,σ). Now, we have x ∈ M ⇒ f (x) ∈ f (M) ⇒ y ∈ f (M) & $f^{-1}(F)$ ⊂ N ⇒ f $[f^{-1}(F)]$ ⊂ f (N)⇒F ⊂ f(N).

Also, (M ∩ N) = ϕ ⇒ f (M) ∩ f (N) = ϕ, as, f is bijection.

Thus, for every point y ∈ Y and each $\mathcal{K}$-closed set F in Y such that y ∉ F, there exist $\mathcal{K}$ - open sets f (M), f (N) in (Y,σ) such that y ∈ f (M), F⊂ f(N) and f (M) ∩ f (N) = ϕ. This claims that (Y,σ) is a $\mathcal{K}$*-regular space.

Hence, the theorem.

Theorem (5.3.2):

(i) Every $\mathcal{K}$ -regular & T_1- space is $\mathcal{K}$ -T_2.

(ii) Every $\mathcal{K}$ *-regular & $\mathcal{K}$ -T_1 space is $\mathcal{K}$ -T_2. (Here $\mathcal{K}$ = p, s, α, b & β).

Proof:

(i) Suppose that (X ,T) is $\mathcal{K}$ -regular and T_1 space. Let x and y be two distinct points in X. Since (X,T) is T_1- space, hence, {x} is closed and y ∉{x}.
Since, (X,T) is $\mathcal{K}$ -regular, hence, there exist disjoint $\mathcal{K}$-open sets U and V in (X,T) containing {x} and y respectively. It follows that X is $\mathcal{K}$ -T_2. Hence, (i).

(ii) Suppose that (X,T) is $\mathcal{K}$*-regular and $\mathcal{K}$ -T_1 space. Let x and y be two distinct points in (X,T). Since X is $\mathcal{K}$-T_1, hence, {x} is $\mathcal{K}$-closed and y∉{x}.Since, (X,T) is $\mathcal{K}$ *-regular, there exist disjoint $\mathcal{K}$-open sets U and V in (X,T) containing {x} and y respectively. It follows that (X ,T)is $\mathcal{K}$ -T_2. Hence, (ii).

Conclusion:

The characterization theorems as well as the preservation theorems for $\mathcal{K}$C-regular, $\mathcal{K}$*-regular & $\mathcal{K}$-regular spaces where $\mathcal{K}$ = p,s,α,b &β with common basic properties have been focused and prepared as a ready reckoner for the research scholars.

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