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A Tri-dimensional Deduction and Application of the Triple Integral Calculation Formula

TANG Guanli

(Yangtze University, School of Information and Mathematics, Jingzhou, Hubei, China, 434023)

QIN Chao

(Wenxing Junior High School, Jingzhou, Hubei, China, 434023)

Abstract: Based on the physical meanings of definite integral, double integral, and triple integral, this paper tries to deduct, in rectangular coordinates, from a tri-dimensional point of view, two triple integral calculation formulas, and provides some application examples.

Keywords: triple integral calculation formulas; physical meanings of multiple integrals; projection method; slicing method

Most textbooks of *Advanced Mathematics* and *Calculus* deduct double integral calculation formulas by calculating the cubic volume of the parallel sectional area, while the deduction of triple integral calculation formula is merely confined to descriptive introduction of the mathematical method, resulting in students' confusion on the actual deduction process of triple integral calculation formula, which makes it rather difficult for them to apply the method with flexibility in actual use. To help students better understand and use the method, the tri-dimensional meanings of the triple integral calculation formulas are given in rectangular coordinates, together with some application examples.

1. The physical meanings of definite integral, double integral, and triple integral

The physical meaning of definite integral: $\int_a^b f(x)dx$ = the inhomogeneous thin line mass $\left(f(x)\geq 0, continuous\right)$ that occupies the interval domain $[a,b]$ of the *x* axis and has a density distribution of $\rho = f(x)$.

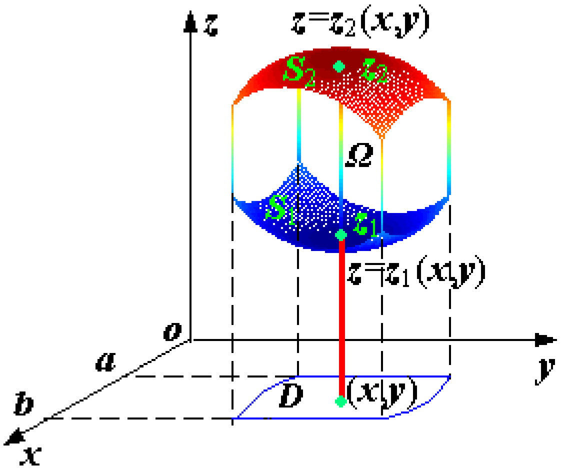
The physical meaning of double integral: the inhomogeneous plane slice mass $\left(f(x)\geq 0, continuous\right)$that occupies the plane domain *D* and has a density distribution of $\rho = f(x,y)$.

The physical meaning of triple integral: the inhomogeneous cubic mass$\left(f(x)\geq 0, continuous\right)$ that occupies the spatial domain Ω and has a density distribution of $\rho = f(x,y,z)$.

2. A tri-dimensional deduction of the triple integral calculation formula

Projection method: suppose there are at most two intersections between Ω and a straight line that parallels the *z* axis and passes through Ω. By projecting Ω onto the *xoy* plane, we have a closed plane domain D. By using the boundary curve of D as the directrix, we can plot a cylinder whose generatrix parallels the z axis, dividing Ω into an upper and a lower surface: $S_1 : z = z_1(x, y)$, $S_2 : z = z_2(x, y)$; $z = z_1(x, y)$, $z = z_2(x, y)$ are continuous in D, and $z_1(x, y) \le z_2(x, y)$. Suppose there is a P meeting the condition $P\ (x, y) \in D$, if we plot a straight line passing the point (x, y) that parallels the z axis, the straight line must pass through Ω, enter from S_1, with $A\ (x, y, z_1(x, y))$ as its intersection coordinates, and exit from S_2, with $B\ (x, y, z_2(x, y))$ as its intersection coordinates.

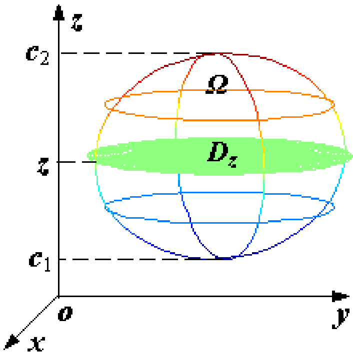
Suppose we project the cubic Ω into a plane slice D, ie mass plane, by vertical compression. Hence the mass of the cubic Ω equals that of the projected plane slice D. In fact, any point mass $P\ (x, y)$ in D is compressed from line segment AB, so its mass equals that of line segment AB, ie $\int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)dz$, which is also the density distribution of the plane slice D, ie $\rho\ (x, y)$. Therefore, the mass of the cubic Ω is that of the projected plane slice D, formulated as $\iint_D \left(\int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)dz \right) dxdy$, which is the formula of the projection method.



Slicing method: suppose the projective interval of the cubic Ω on z axis is $[c_1, c_2]$, marked as $C(0,0,c_1)$ and $D(0,0,c_2)$, suppose there is a z meeting the condition $z \in [c_1, c_2]$, if we plot a plane, passing the point $P\ (0,0,z)$, perpendicular to the z axis, passing through Ω, we will get a cross section of Ω, marked as D_z.

If the cubic Ω is projected into a line segment CD by vertical compression, the mass of the cubic

Ω equals that of the projected line segment CD, and any point mass $P\left(0,0,z\right)$ in CD is compressed from the cross section D_z, so the mass of P equals that of D_z, ie $\iint\limits_{D_z} f\left(x,y,z\right)dxdy$, which is also the density distribution $\rho\left(z\right)$ of the line segment CD. Therefore, the mass of the cubic Ω is that of the projected line segment CD, formulated as $\int_{c_1}^{c_2}\left(\iint\limits_{D_z} f\left(x,y,z\right)dxdy\right)dz$, which is the formula of the slicing method.



3. Examples

Example 1. Calculate the triple integral $\iiint\limits_{\Omega} xdxdydz$, Ω comprised of three coordinate planes and a plane of $x+y+z=1$.

Solution: Projection method: project Ω onto the plane xoy, and we get the projected domain:

$$D: 0 \le x \le 1, 0 \le y \le 1-x$$

Then we have: $\iiint\limits_{\Omega} xdxdydz = \iint\limits_{D}\left(\int_0^{1-x-y} xdz\right)dxdy = \int_0^1 dx\int_0^{1-x} xdy\int_0^{1-x-y} dz = \frac{1}{24}$

Example 2. Calculate the triple integral $\iiint\limits_{\Omega} z^2dxdydz$, Ω comprised of a surface $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Solution: Slicing method: the projection of Ω on axis z is $\left[-c,c\right]$, so $D_z=\left\{\left(x,y\right)/\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1-\frac{z^2}{c^2}\right\}$.

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Then we have: $$\iiint_{\Omega} z^2 dxdydz = \int_{-c}^{c}\left(\iint_{D_z} z^2 dxdy\right)dz = \int_{-c}^{c}\left(z^2 \pi ab\left(1-\frac{z^2}{c^2}\right)\right)dz = \frac{4}{15}\pi abc^3$$

As we can see, projection method applies to example 1, because: first, D_z is difficult to present; second, the integrand consists of *x*, which makes it time-consuming to first integrate *xy*, but *z* is the opposite. Slicing method applies to example 2, because: first, D_z is easy to present; second, the integrand consists of *z*, which makes it rather time-consuming and difficult to first integrate *z*, but *xy* is the opposite.

4. Conclusion

Based on the physical meanings of definite integral, double integral, and triple integral, this paper tries to deduct, in rectangular coordinates, from a tri-dimensional point of view, two calculation formulas of triple integrals, and the tri-dimensional meanings of them are presented, followed by two application examples. The question types that the two formulas apply to are also briefed in order to help readers better understand and use the formulas to calculate triple integrals.